Math 170E

Lecture Notes Section 4.5 [*,†]

Bivariate normal distibutions

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NOTE: Materials that appear in the textbook but do not appear in the lecture notes might still be tested. Please send me an email if you find typos.

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[†]This notes is based on March Boedihardjo's and Jamie Haddock's notes from the past quarters, and I would like to thank them for their generosity. "*Nanos gigantum humeris insidentes* (I am but a dwarf standing on the shoulders of giants)".

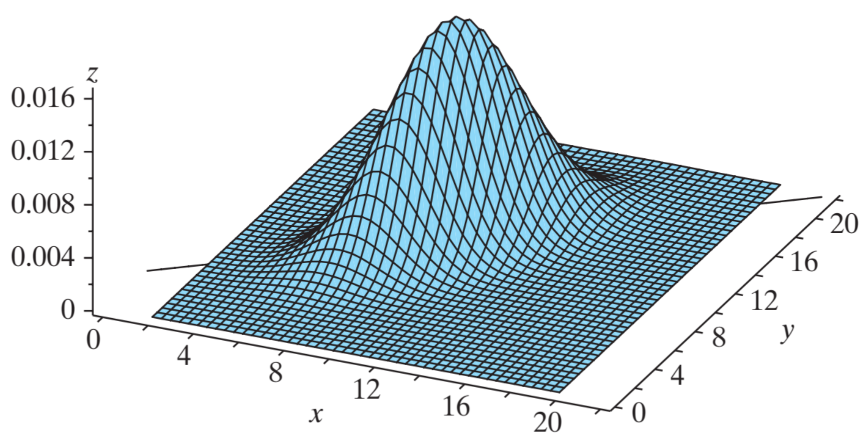
1 Bivariate standard normal RVs

Recall that a standard normal RV X has pdf given by

$$f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{x^2}{2}\right].$$

Let X and Y be two **independent** standard normal RVs. Then their joint pdf is

$$\begin{aligned} f(x,y) = f_X(x) f_Y(y) =& \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{x^2}{2}\right] \cdot \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{y^2}{2}\right] \\ =& \frac{1}{2\pi} \exp\left[\frac{-1}{2}\left(x^2+y^2\right)\right]. \end{aligned}$$



What happens when X and Y are related?

Recall the correlation coefficient of X and Y,

$$\rho = \frac{E[XY] - E[X]\,E[Y]}{\sigma_X \sigma_Y}.$$

Note that $\rho = 0$ when X and Y are independent.

However, when ρ is not equal to 0, what should be the joint pdf for X and Y?

2 Bivariate standard normal: Joint pdfs

Theorem 1. *Let X and Y be standard normal random variables with correlation coefficient ρ.*
The joint pdf of X and Y are then given by

$$f(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left[\frac{-1}{2(1-\rho^2)}\left(x^2 - 2\rho xy + y^2\right)\right].$$

In general, using this joint pdf to compute the probabilities for X and Y is very hard as there is no nice formulas to compute the integral.
So we will use the next theorem to make things easier when looking at the conditional standard normal RVs.

3 Bivariate standard normal: Conditional RVs

Theorem 2. *Let X and Y be bivariate standard normal random variables with correlation coefficient ρ.*

For any $\boldsymbol{x}$, the random variable $Y \mid \boldsymbol{x}$ (i.e., Y given that $X = \boldsymbol{x}$) is a normal RV, with mean

$$E[Y \mid \boldsymbol{x}] \;=\; \rho\, \boldsymbol{x} \qquad \textit{(which is a linear function of } \boldsymbol{x}\textit{)},$$

and with variance

$$\sigma^2_{Y|\boldsymbol{x}} \;=\; (1-\rho^2) \qquad \textit{(which does not depend on } \boldsymbol{x}\textit{)}.$$

4 Bivariate standard normal: Example

Let X and Y be bivariate standard normal RVs with correlation coefficient $\rho = \frac{3}{5}$. Compute

$$P(0.83 < Y < 2.18) \quad \text{and} \quad P(0 < Y < 2 \mid X = \mathbf{\frac{5}{3}}).$$

Answer: For the first part,

$$P(0.83 < Y < 2.18) = \Phi(2.18) - \Phi(0.83)$$

$$= (0.9854) - (0.7967) = 0.1887.$$

For the second part, we have by Theorem 2 that $Y \mid \mathbf{\frac{5}{3}}$ has mean $\rho\,\mathbf{x} = 1$ and variance $1 - \rho^2 = \frac{16}{25}$.

Let Z be the random variable

$$Z \;=\; \frac{(Y \mid \mathbf{x}) - \mu_{Y|\mathbf{x}}}{\sigma_{Y|\mathbf{x}}} \;=\; \frac{(Y \mid \mathbf{\frac{5}{3}}) - 1}{\frac{4}{5}},$$

which is the standard normal RV. Now note that

$$(Y \mid \mathbf{\tfrac{5}{3}}) = 0 \qquad \Rightarrow \qquad Z = \frac{(0) - 1}{\frac{4}{5}} = -\frac{5}{4} = -1.25$$

$$(Y \mid \mathbf{\tfrac{5}{3}}) = 2 \qquad \Rightarrow \qquad Z = \frac{(2) - 1}{\frac{4}{5}} = \frac{5}{4} = 1.25.$$

Hence

$$P(0 < Y < 2 \mid X = \mathbf{\tfrac{5}{3}}) \;=\; P(-1.25 < Z < 1.25)$$

$$=\; \Phi(1.25) - \Phi(-1.25) \;=\; 2\,\Phi(1.25) - 1$$

$$=\; 2(0.8944) - 1 \;=\; 0.7888.$$

5 Bivariate general normal RVs

Recall that the normal RV X with mean μ_X and variance σ_X^2 has pdf

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_X} \exp\left[-\frac{(x-\mu_X)^2}{2\sigma_X^2}\right].$$

Let X and Y be two **independent** normal RVs with their own mean and variance. Then their joint pdf is

$$\begin{aligned} f(x,y) &= f_X(x) f_Y(y) \\ &= \frac{1}{\sqrt{2\pi}\sigma_X} \exp\left[-\frac{(x-\mu_X)^2}{2\sigma_X^2}\right] \frac{1}{\sqrt{2\pi}\sigma_Y} \exp\left[-\frac{(y-\mu_Y)^2}{2\sigma_Y^2}\right] \\ &= \frac{1}{2\pi\sigma_X\sigma_Y} \exp\left[\frac{-1}{2}\left(\left[\frac{x-\mu_X}{\sigma_X}\right]^2 + \left[\frac{y-\mu_Y}{\sigma_Y}\right]^2\right)\right]. \end{aligned}$$

What happens when X and Y are related (i.e., when $\rho \neq 0$)? What should be the joint pdf for X and Y?

6 Bivariate general normal RVs: Joint pdfs

Theorem 3. *Let X and Y be general normal random variables with correlation coefficient ρ.*
The joint pdf of X and Y are then given by

$$f(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left[-\frac{q(x,y)}{2}\right],$$

where

$$q(x,y) = \frac{1}{(1-\rho^2)}\left(\left[\frac{x-\mu_X}{\sigma_X}\right]^2 - 2\rho\left[\frac{x-\mu_X}{\sigma_X}\right]\left[\frac{y-\mu_Y}{\sigma_Y}\right] + \left[\frac{y-\mu_Y}{\sigma_Y}\right]^2 \right).$$

Again, it is not advisable to use this joint pdf to compute probabilities. So we will use the following theorem to make things easier when looking at the conditional general random variables.

7 Bivariate general normal: Conditional RVs

Theorem 4. *Let X and Y be bivariate general normal RVs with correlation coefficient ρ.*

For any $\boldsymbol{x}$, the random variable $Y \mid \boldsymbol{x}$ is a normal RV, with mean

$$E[Y \mid \boldsymbol{x}] = \mu_Y + \rho \frac{\sigma_Y}{\sigma_X}(\boldsymbol{x} - \mu_X) \qquad \text{(a linear function of } \boldsymbol{x}\text{)},$$

and variance

$$\sigma^2_{Y|\boldsymbol{x}} = \sigma^2_Y (1 - \rho^2) \qquad \text{(does not depend on } \boldsymbol{x}\text{)}.$$

8 Bivariate general normal RVs: Example

Let X and Y be bivariate normal RVs with parameters $\mu_X = 2.9$, $\mu_Y = 2.4$, $\sigma_X = 0.4$, $\sigma_Y = 0.5$, and $\rho = 0.8$. Compute the probability

1. $P(2.1 < Y < 3.3)$.
2. $P(2.1 < Y < 3.3 \mid X = \mathbf{3.2})$

Answer: For the first part, let $Z = \frac{Y-2.4}{0.5}$. Note that Z is a standard normal random variable. So we have

$$P(2.1 < Y < 3.3) \;=\; P\left(\frac{2.1-2.4}{0.5} < Z < \frac{3.3-2.4}{0.5}\right)$$

$$=\; P(-0.6 < Z < 1.8) \;=\; \Phi(1.8) - \Phi(-0.6)$$

$$=\; 0.9641 - 0.2743 \;=\; 0.6898.$$

For the second part, using Theorem 4 we have $Y \mid \mathbf{3.2}$ is normal RV with mean and variance

$$\mu_{Y|\mathbf{3.2}} = 2.4 + (0.8)\frac{(0.5)}{(0.4)}(3.2 - 2.9) \;=\; 2.7;$$
$$\sigma^2_{Y|\mathbf{3.2}} = (0.5)^2(1 - (0.8)^2) = (0.5)^2(0.6)^2 \;=\; (0.3)^2.$$

Now, let $Z = \frac{(Y|3.2)-2.7}{0.3}$. Then Z is a standard normal random variable. So we have

$$\begin{aligned}
&P(2.1 < Y < 3.3 \mid X = \mathbf{3.2})\\
&= \; P\left(\frac{2.1 - 2.7}{0.3} < Z < \frac{3.3 - 2.7}{0.3}\right)\\
&= \; P(-2 < Z < 2) \;=\; \Phi(2) - \Phi(-2)\\
&= \; 0.9772 - 0.0228 \;=\; 0.9544.
\end{aligned}$$

9 Bivariate general normal RVs: Example 2

Let X and Y have a bivariate normal distribution.

(In particular, $E[Y \mid \mathbf{x}]$ is a line!)

Find two different lines, $a(\mathbf{x})$ and $b(\mathbf{x})$, parallel to and equidistant from $E[Y \mid \mathbf{x}]$, such that

$$P[a(\mathbf{x}) < Y < b(\mathbf{x}) \mid X = \mathbf{x}] = 0.9544 \qquad \text{for all real } \mathbf{x}.$$

10 Bivariate general normal RVs: Answer 2

First note that $E[Y \mid \mathbf{x}]$ is equal to

$$\begin{aligned} E[Y \mid \mathbf{x}] &= \mu_Y + \rho\left(\frac{\sigma_Y}{\sigma_X}\right)(\mathbf{x} - \mu_X) \\ &= -1 + \left(\frac{3}{5}\right)\left(\frac{5}{3}\right)(\mathbf{x} - 2) \;=\; \mathbf{x} - 3. \end{aligned}$$

This is the line equation for $E[Y \mid \mathbf{x}]$. (Draw the picture.) Since $a(\mathbf{x})$ and $b(\mathbf{x})$ are two lines that are parallel and equidistant from $E[Y \mid \mathbf{x}]$, their line equations are

$$\begin{aligned} a(\mathbf{x}) &= \mathbf{x} - 3 - c; \\ b(\mathbf{x}) &= \mathbf{x} - 3 + c; \end{aligned}$$

for some $c > 0$.

Now note that $Y \mid \mathbf{x}$ has mean and variance

$$\mu_{Y|\mathbf{x}} = E[Y \mid \mathbf{x}] = \mathbf{x} - 3;$$
$$\sigma_{Y|\mathbf{x}} = \sigma_Y \sqrt{1-\rho^2} \; = \; (5)\sqrt{1-(3/5)^2} \; = \; 4.$$

Let $Z = \frac{(Y|\mathbf{x}) - (\mathbf{x} - 3)}{4}$. Then Z is the standard normal random variable. So we have

$$\begin{aligned}
& P[a(\mathbf{x}) < Y < b(\mathbf{x}) \mid X = \mathbf{x}] \\
= \; & P\left(\frac{a(\mathbf{x}) - (\mathbf{x} - 3)}{4} < Z < \frac{b(\mathbf{x}) - (\mathbf{x} - 3)}{4}\right) \\
= \; & P\left(\frac{-c}{4} < Z < \frac{c}{4}\right) \; = \; \Phi\left(\frac{c}{4}\right) - \Phi\left(-\frac{c}{4}\right) \\
= \; & 2\,\Phi\left(\frac{c}{4}\right) - 1.
\end{aligned}$$

Since the question gives us $P[a(\mathbf{x}) < Y < b(\mathbf{x}) \mid X = \mathbf{x}] = 0.9544$, we then have

$$\begin{aligned} 0.9544 &= 2\,\Phi\left(\frac{c}{4}\right) - 1 \\ \Phi\left(\frac{c}{4}\right) &= \frac{(0.9544)+1}{2} \\ \Phi\left(\frac{c}{4}\right) &= 0.9772. \end{aligned}$$

By using Table Va in Appendix B, so we have

$$\frac{c}{4} = 2.$$

Hence $c = 8$. Therefore $a(\mathbf{x})$ and $b(\mathbf{x})$ are the lines

$$a(\mathbf{x}) = \mathbf{x} - 11; \qquad b(\mathbf{x}) = \mathbf{x} + 5.$$

(Draw the picture.)