Math 170E

Lecture Notes Section 4.4 [*,†]

Bivariate distibutions of continuous type

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NOTE: Materials that appear in the textbook but do not appear in the lecture notes might still be tested. Please send me an email if you find typos.

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[†]This notes is based on March Boedihardjo's and Jamie Haddock's notes from the past quarters, and I would like to thank them for their generosity. "*Nanos gigantum humeris insidentes* (I am but a dwarf standing on the shoulders of giants)".

1 Bivariate RVs of continuous type

Let X and Y be two RVs of continuous type.

The **joint probability density function** $f(\cdot, \cdot)$ of X and Y is the function such that

$$P(X \leq a, Y \leq b) \quad = \quad \int_{-\infty}^{b} \int_{-\infty}^{a} f(x, y)\, dx\, dy.$$

That is, we extend the idea of joint distribution from Section 4.1 to continuous type, by substituting summations with integrals.

2 Bivariate continuous RVs: Example

Let X and Y be independent uniform RV from $[0, 1]$.

The joint pdf of X and Y are given by

$$f(x, y) = \begin{cases} 1 & 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1; \\ 0 & \text{otherwise.} \end{cases}$$

Indeed, suppose that $a, b \in [0, 1]$. Then

$$P(X \leq a, Y \leq b) = P(X \leq a)P(Y \leq b) = ab;$$

$$\int_{-\infty}^{b} \int_{-\infty}^{a} f(x, y)\, dx\, dy = \int_{0}^{b} \int_{0}^{a} 1\, dx\, dy = \int_{0}^{b} a\, dy = ab.$$

Check $P(X \leq a, Y \leq b) = \int_{-\infty}^{b} \int_{-\infty}^{a} f(x, y)\, dx\, dy$ for other values of a and b as an exercise.

3 Bivariate continuous RVs: Properties

Theorem 1. *The joint pdf $f(\cdot,\cdot)$ of X and Y satisfies*

- $f(x, y) \geq 0$;
- $f(x, y) = 0$ *when (x, y) is not in the support S of X and Y;*
- $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\, dx\, dy = 1$;
- *For any region A of the plane,*

$$P[(X, Y) \in A] \quad = \quad \int_A f(x, y)\, dx\, dy\,.$$

4 Bivariate continuous RVs: Example

Let X and Y be RVs with the joint pdf

$$f(x, y) = \frac{4}{3}(1 - xy) \qquad \text{for } 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1.$$

Compute the probabilities

$$P(X \leq 1/2, Y \leq 1/4); \qquad P(Y \leq X/2)$$

5 Bivariate continuous RVs: Answer

For the first probability, (BT)

$$P(X \le 1/2,\, Y \le 1/4) \;=\; \int_0^{1/4} \int_0^{1/2} \frac{4}{3}(1 - xy)\; dx\, dy$$

$$= \int_0^{1/4} \left[\frac{4}{3}\left(x - \frac{x^2 y}{2} \right) \right]_{x=0}^{x=1/2} dy \;=\; \int_0^{1/4} \left[\frac{4}{3}\left(\frac{1}{2} - \frac{y}{8} \right) \right] dy$$

$$= \left[\frac{4}{3}\left(\frac{y}{2} - \frac{y^2}{16} \right) \right]_0^{1/4} \;=\; \left[\frac{4}{3}\left(\frac{1}{8} - \frac{1}{256} \right) \right] \;=\; \frac{31}{192}.$$

For the second probability, (draw picture)

$$P(Y \le X/2) \;=\; \int_0^{1} \int_0^{x/2} \frac{4}{3}(1 - xy)\; dy\, dx$$

$$= \int_0^{1} \left[\frac{4}{3}\left(y - \frac{xy^2}{2} \right) \right]_{y=0}^{y=x/2} dx \;=\; \int_0^{1} \left[\frac{4}{3}\left(\frac{x}{2} - \frac{x^3}{8} \right) \right] dx$$

$$= \left[\frac{4}{3}\left(\frac{x^2}{4} - \frac{x^4}{32} \right) \right]_0^{1} \;=\; \left[\frac{4}{3}\left(\frac{1}{4} - \frac{1}{32} \right) \right] \;=\; \frac{7}{24}.$$

6 Marginal pdfs: Formula

The (marginal) probability density function of X is

$$f_X(\mathbf{x}) = \int_{-\infty}^{\infty} f(\mathbf{x}, y)\, dy \qquad \text{for any } \mathbf{x} \in S_X.$$

The (marginal) probability density function of Y is

$$f_Y(\mathbf{y}) = \int_{-\infty}^{\infty} f(x, \mathbf{y})\, dx \qquad \text{for any } \mathbf{y} \in S_Y.$$

7 Marginal pdfs: Example

Let X and Y be as in the previous example.

The pdf of X is then given by

$$\begin{aligned} f_X(x) &= \int_0^1 \frac{4}{3}(1 - xy)\, dy = \left[\frac{4}{3}\left(y - \frac{xy^2}{2}\right)\right]_0^1 \\ &= \frac{4}{3}\left(1 - \frac{x}{2}\right) \qquad \text{for } 0 \le x \le 1. \end{aligned}$$

As an exercise, show that the pdf of Y is

$$f_Y(y) = \frac{4}{3}\left(1 - \frac{y}{2}\right) \qquad \text{for } 0 \le y \le 1.$$

8 Mean and variance of continuous bivariate RVs

The mean of X is given by

$$\mu_X \;=\; E[X] \;=\; \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\, f(x,y)\, dx\, dy.$$

The variance of X is given by

$$\begin{aligned}\sigma_X^2 \;&=\; E[(X-\mu_X)^2] \;=\; \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x-\mu_X)^2\, f(x,y)\, dx\, dy \\ &=\; E[X^2]-(\mu_X)^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2\, f(x,y)\, dx\, dy - (\mu_X)^2.\end{aligned}$$

9 Expectations of continuous bi-RVs: Example

Let X and Y be the example from before.

The mean of X is (BT)

$$
\begin{aligned}
\mu_X &= \int_0^1 \int_0^1 x\,\frac{4}{3}\,(1 - xy)\;dy\,dx \\
&= \int_0^1 \left[\frac{4x}{3}\left(y - \frac{xy^2}{2}\right)\right]_{y=0}^{y=1} dx \;=\; \int_0^1 \left[\frac{4}{3}\left(x - \frac{x^2}{2}\right)\right] dx \\
&= \left[\frac{4}{3}\left(\frac{x^2}{2} - \frac{x^3}{6}\right)\right]_0^1 \;=\; \frac{2}{3} - \frac{2}{9} \;=\; \frac{4}{9}.
\end{aligned}
$$

The second moment of X is

$$
\begin{aligned}
E[X^2] &= \int_0^1 \int_0^1 x^2\,\frac{4}{3}\,(1 - xy)\;dy\,dx \\
&= \int_0^1 \left[\frac{4x^2}{3}\left(y - \frac{xy^2}{2}\right)\right]_{y=0}^{y=1} dx \;=\; \int_0^1 \left[\frac{4}{3}\left(x^2 - \frac{x^3}{2}\right)\right] dx \\
&= \left[\frac{4}{3}\left(\frac{x^3}{3} - \frac{x^4}{8}\right)\right]_0^1 \;=\; \frac{4}{9} - \frac{4}{24} \;=\; \frac{5}{18}.
\end{aligned}
$$

The variance of X is thus given by

$$\sigma_X^2 \;=\; E[X^2] - (\mu_X)^2 \;=\; \frac{5}{18} - \frac{16}{81} \;=\; \frac{13}{162}.$$

We can also compute the mean and variance of X using the pdf of X directly. Try to do it as an exercise.

10 Expectations of continuous bi-RVs: Example 2

Let X and Y have the joint pdf

$$f(x,y) = 4, \qquad 0 \le x \le 1,\ 0 \le y \le x/2.$$

Note that the support $S = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le x/2\}$ is a triangle (see picture).

The probability $P(0 \le X \le 1/2,\ 0 \le Y \le 1/2)$ is (draw picture)

$$P(0 \le X \le 1/2,\ 0 \le Y \le 1/2) \ = \ \int_0^{1/2} \int_0^{x/2} 4\, dy\, dx$$

$$= \ \int_0^{1/2} [4y]_{y=0}^{y=x/2}\, dx \ = \ \int_0^{1/2} 2x\, dx \ = \ \frac{1}{4}.$$

The marginal pdf of X is (draw picture)

$$f_X(x) \;=\; \int_0^{x/2} 4\,dy \;=\; 2x \qquad \text{with support} \quad 0 \le x \le 1.$$

The marginal pdf of Y is (draw picture)

$$\begin{aligned} f_Y(y) \;&=\; \int_{2y}^{1} 4\,dx \;=\; [4x]_{2y}^{1} \;=\; 4(1) - 4(2y) \\ &=\; 4 - 8y \qquad \text{with support} \quad 0 \le y \le 1/2. \end{aligned}$$

The mean of X is (draw picture)

$$\mu_X = \int_0^1 \int_0^{x/2} x(4)\, dy\, dx = \int_0^1 [4xy]_{y=0}^{y=x/2}\, dx$$

$$= \int_0^1 2x^2\, dx = \left[\frac{2}{3}x^3\right]_0^1 = \frac{2}{3}.$$

The mean of Y is (draw picture)

$$\mu_Y = \int_0^{1/2} \int_{2y}^{1} y(4)\, dx\, dy = \int_0^{1/2} [4xy]_{x=2y}^{x=1}\, dy$$

$$= \int_0^{1/2} 4y - 8y^2\, dx = \left[2y^2 - \frac{8y^3}{3}\right]_0^{1/2} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

11 Independent joint pdfs

Theorem 2. *X and Y are independent if and only if*

$$f(x, y) = f_X(x)\, f_Y(y) \qquad x \in S_X, y \in S_y.$$

Compare this to the analogous result for joint pmfs in Section 4.2.

12 Independent joint pdfs: Example

Suppose $f(x, y) = 1$ for $0 \le x \le 1$ and $0 \le y \le 1$.

The pdf of X and Y are given by

$$f_X(x) = \begin{cases} \int_0^1 1\,dy = 1 & \text{for all } 0 \le x \le 1; \\ \int_0^1 0\,dy = 0 & \text{for } x \le 0 \text{ or } x \ge 1. \end{cases}$$

$$f_Y(y) = \begin{cases} \int_0^1 1\,dx = 1 & \text{for all } 0 \le y \le 1; \\ \int_0^1 0\,dx = 0 & \text{for } y \le 0 \text{ or } y \ge 1. \end{cases}$$

We now have

$$f(x, y) = f_X(x)\, f_Y(y) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \text{ and } 0 \le y \le 1; \\ 0 & \text{otherwise.} \end{cases}$$

So X and Y are independent.

13 Dependent joint pdfs: Example

Let X and Y be RVs with the joint pdf

$$f(x,y) = \frac{4}{3}(1 - xy) \qquad \text{for } 0 \le x \le 1 \text{ and } 0 \le y \le 1.$$

We have computed their marginal pdfs,

$$\begin{aligned} f_X(x) &= \frac{4}{3}\left(1 - \frac{x}{2}\right) \qquad \text{for } 0 \le x \le 1; \\ f_Y(y) &= \frac{4}{3}\left(1 - \frac{y}{2}\right) \qquad \text{for } 0 \le y \le 1. \end{aligned}$$

Consider $x = 1/4$ and $y = 1/8$. We then have

$$f(1/4, 1/4) \;=\; \frac{4}{3}\left(1 - \frac{1}{4} \times \frac{1}{4}\right) \;=\; \frac{5}{4}.$$

On the other hand, for the marginal pdfs,

$$f_X(1/4) \;=\; \frac{4}{3}\left(1 - \frac{1}{2} \times \frac{1}{4}\right) \;=\; \frac{7}{6}$$

$$f_Y(1/4) \;=\; \frac{4}{3}\left(1 - \frac{1}{2} \times \frac{1}{4}\right) \;=\; \frac{7}{6}.$$

So

$$f_X(x)\, f_Y(y) \;=\; \frac{49}{36}.$$

Therefore X and Y are not independent.