Math 170E

Lecture Notes Section 4.3 [*†]

Conditional distributions

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NOTE: Materials that appear in the textbook but do not appear in the lecture notes might still be tested. Please send me an email if you find typos.

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[†]This notes is based on March Boedihardjo's and Jamie Haddock's notes from the past quarters, and I would like to thank them for their generosity. "*Nanos gigantum humeris insidentes* (I am but a dwarf standing on the shoulders of giants)".

1 Conditional probability: recap

Recall that, the probability of the event A given that event B has happened is

$$P(A \mid B) \quad = \quad \frac{P(A \cap B)}{P(B)}.$$

2 Conditional probability: bivariate RVs

Let X and Y be bivariate random variables with joint pmf $f(\cdot,\cdot)$.

Let $A = \{X = x\}$ and $B = \{Y = y\}$. Then

$$\begin{aligned} P(X = x \mid Y = y) &= \frac{P(A \cap B)}{P(B)} \\ &= \frac{P(X = x, Y = y)}{P(Y = y)} \\ &= \frac{f(x, y)}{f_Y(y)}. \end{aligned}$$

3 Conditional probability: Example

Toss a fair coin twice.

Let X be the smaller of the two coin flips and Y the larger of the two coin flips.

Let $f(\cdot,\cdot)$ be the joint pmf of X and Y. Recall

$$
\begin{aligned}
f(0,0) &= P(\text{1st coin is 0, 2nd coin is 0}) \;=\; \frac{1}{4};\\
f(0,1) &= P(\text{1st coin is 0, 2nd coin is 1})\\
&\qquad\quad + P(\text{1st coin is 1, 2nd coin is 0}) \;=\; \frac{1}{2};\\
f(1,0) &= 0;\\
f(1,1) &= P(\text{1st coin is 1, 2nd coin is 1}) \;=\; \frac{1}{4};
\end{aligned}
$$

Then

$$P(X = 0 \mid \mathbf{Y{=}1}) = \frac{f(0,1)}{f_Y(1)} = \frac{f(0,1)}{f(0,1) + f(1,1)} = \frac{1/2}{3/4} = \frac{2}{3};$$
$$P(X = 1 \mid \mathbf{Y{=}1}) = \frac{f(1,1)}{f_Y(1)} = \frac{f(1,1)}{f(0,1) + f(1,1)} = \frac{1/4}{3/4} = \frac{1}{3}.$$

The important takeaway here is that X **given** $Y = 1$ is **another random variable**!

4 Conditional RV: Definition

Fix an element $\mathbf{y}$ so that $f_Y(\mathbf{y}) > 0$. The random variable X given $Y = \mathbf{y}$ has the probability mass function $g(\cdot \mid \mathbf{y})$,

$$g(x \mid \mathbf{y}) = \frac{f(x, \mathbf{y})}{f_Y(\mathbf{y})}.$$

The function $g(\cdot \mid \mathbf{y})$ is called the **conditional probability mass function** of X given $Y = \mathbf{y}$.

The random variable X given $Y = \mathbf{y}$ is usually denoted $X \mid (Y = \mathbf{y})$ or $X \mid \mathbf{y}$.

Analogously, fix an element $\mathbf{x}$ so that $f_X(\mathbf{x}) > 0$. The random variable Y given $X = \mathbf{x}$ has the probability mass function $h(\cdot \mid \mathbf{x})$,

$$h(y \mid \mathbf{x}) = \frac{f(\mathbf{x}, y)}{f_X(\mathbf{x})}.$$

5 Conditional RV: Example

Let X and Y be the smaller and the larger of two coin tosses, respectively.

The random variable $X \mid (Y = \mathbf{0})$ has pmf

$$g(0 \mid \mathbf{0}) = 1; \qquad g(1 \mid \mathbf{0}) = 0.$$

So $X \mid (Y = \mathbf{0})$ is a random variable that is always equal to 0.

In particular, $g(\cdot \mid \mathbf{0})$ is also a pmf of an RV.

The random variable $X \mid (Y = \mathbf{1})$ has pmf

$$g(0 \mid \mathbf{1}) \quad = \quad \frac{2}{3}; \qquad g(1 \mid \mathbf{1}) \quad = \quad \frac{1}{3}.$$

So $X \mid (Y = \mathbf{1})$ is an RV for which 0 is twice as likely as 1.

In particular, $g(\cdot \mid \mathbf{1})$ is also a pmf of an RV.

Compute $Y \mid (X = \mathbf{0})$ and $Y \mid (X = \mathbf{1})$ as exercises.

6 Conditional expectations: Definition

Conditional mean of X given $Y = \mathbf{y}$, is

$$\mu_{X|\mathbf{y}} \quad = \quad E[X \mid \mathbf{y}] \quad = \quad \sum_{x \in S_X} x\, g(x \mid \mathbf{y}).$$

Conditional variance of X given $Y = \mathbf{y}$, is

$$\begin{aligned} \sigma^2_{X|\mathbf{y}} &= E[(X - E[X \mid \mathbf{y}])^2) \mid \mathbf{y}] \\ &= \sum_{x \in S_X} (x - \mu_{X|\mathbf{y}})^2\, g(x \mid \mathbf{y}) \\ &= E[X^2 \mid \mathbf{y}] - (\mu_{X|\mathbf{y}})^2. \end{aligned}$$

7 Conditional expectations: Example

Let X and Y be as in coin-tosses example.

For $X \mid (Y = \mathbf{0})$, we have

$$\begin{aligned}
\mu_{X|\mathbf{0}} &= 0\, g(0 \mid \mathbf{0}) + 1\, g(1 \mid \mathbf{0}) \;=\; 0 \times 1 \,+\, 1 \times 0 \;=\; 0; \\
\sigma^2_{X|\mathbf{0}} &= (0-0)^2\, g(0 \mid \mathbf{0}) + (1-0)^2\, g(1 \mid \mathbf{0}) \\
&= 0 \times 1 \,+\, 1 \times 0 \;=\; 0.
\end{aligned}$$

For $X \mid (Y = \mathbf{1})$, we have

$$\begin{aligned}
\mu_{X|\mathbf{1}} &= 0\, g(0 \mid \mathbf{1}) + 1\, g(1 \mid \mathbf{1}) \;=\; 0 \times \frac{2}{3} + 1 \times \frac{1}{3} \;=\; \frac{1}{3}; \\
\sigma^2_{X|\mathbf{1}} &= \left(0 - \frac{1}{3}\right)^2 g(0 \mid \mathbf{1}) + \left(1 - \frac{1}{3}\right)^2 g(1 \mid \mathbf{1}) \\
&= \frac{1}{9} \times \frac{2}{3} \,+\, \frac{4}{9} \times \frac{1}{3} \;=\; \frac{6}{27} \;=\; \frac{2}{9}.
\end{aligned}$$

8 Trinomial distributions: Recap

Recall that X and Y with parameter n, p_X, p_Y are

- We have n experiments, each experiment is a "success" with probability p_X, "half-success" with probability p_Y, and "failure" with probability $1 - p_X - p_Y$.
- X is the number of successful experiments;
- Y is the number of half-successful experiments.

9 Trinomial distributions: Example

In a class of 40 students, a student can do very well with probability 0.3, do reasonably well with probability 0.6, and do badly with probability 0.1.

Describe the cdf of the number of students that do reasonably well, given that number of students that do very well is 11.

10 Trinomial distribution: Answer

Let X be the number of students that do very well, and let Y be the number of students that do reasonably well. Then X and Y are trinomial random variables with $n = 40$, $p_X = 0.3$ and $p_Y = 0.6$.

The question is asking for the cdf of $Y \mid (X = \mathbf{11})$.

Given that the number of students that do very well is **11**, the rest of the 29 students can only do reasonably well or do badly.

So $Y \mid (X = \mathbf{11})$, is a binomial random variable with $29 = 40 - \mathbf{11}$ experiments.

What is the (half)-success probability and the failure probability of $Y \mid (X = \mathbf{11})$?

Before conditioning, we know that doing reasonably well is 6 times as likely as doing badly. This should not change after conditioning!

The (half)-success probability of $Y \mid (X = \mathbf{11})$ is thus

$$\frac{6}{7} \quad = \quad \frac{(0.6)}{1 - (0.3)} \quad = \quad \frac{p_Y}{1 - p_X}.$$

The failure probability of $Y \mid (X = \mathbf{11})$ is thus

$$\frac{1}{7} \quad = \quad \frac{1 - (0.3) - (0.6)}{1 - (0.3)} \quad = \quad \frac{1 - p_X - p_Y}{1 - p_X}.$$

11 Trinomial distributions: Conditional version

Let X and Y be trinomial random variables with parameter n, p_X, p_Y.

Then, $X \mid (Y = \mathbf{y})$ is the binomial random variable with parameter $n - \mathbf{y}$, success probability $\frac{p_X}{1-p_Y}$ and failure probability $\frac{1-p_X-p_Y}{1-p_Y}$;

Also, $Y \mid (X = \mathbf{x})$ is the binomial random variable with parameter $n-\mathbf{x}$, half success probability $\frac{p_Y}{1-p_X}$ and failure probability $\frac{1-p_X-p_Y}{1-p_X}$;

The correlation coefficient of X and Y is given by

$$\rho \quad = \quad -\sqrt{\frac{p_X p_Y}{(1-p_X)(1-p_Y)}}.$$