Math 170E

Lecture Notes Section 4.2 *†

Correlation coefficient

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NOTE: Materials that appear in the textbook but do not appear in the lecture notes might still be tested. Please send me an email if you find typos.

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†This notes is based on March Boedihardjo's and Jamie Haddock's notes from the past quarters, and I would like to thank them for their generosity. "*Nanos gigantum humeris insidentes* (I am but a dwarf standing on the shoulders of giants)".

1 Covariance and correlation coefficient

Let X and Y be two random variables.

The **covariance** of X and Y is

$$\sigma_{XY} = E[(X-\mu_X)(Y-\mu_Y)] = E[XY]-\mu_X\mu_Y.$$

The covariance is also denoted by $\mathrm{Cov}(X,Y)$.

The **correlation coefficient** of X and Y is given by

$$\rho = \frac{\sigma_{XY}}{\sigma_X\sigma_Y}.$$

Note: ρ is always between -1 and 1.

2 Correlation coefficient: Independent example

Let X and Y be two independent coin tosses. We have

$$
\begin{aligned}
E[XY] &= \sum_{(x,y)\in S} x\,y\,f(x,y) \\
&= (0)(0)f(0,0) + (0)(1)f(0,1) + (1)(0)f(1,0) + (1)(1)f(1,1) \\
&= (0)(0)\left(\frac{1}{4}\right) + (0)(1)\left(\frac{1}{4}\right) + (1)(0)\left(\frac{1}{4}\right) + (1)(1)\left(\frac{1}{4}\right) \\
&= \frac{1}{4}.
\end{aligned}
$$

The covariance is then given by

$$\sigma_{XY} = E[XY] - \mu_X\mu_Y = \frac{1}{4} - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = 0.$$

The correlation coefficient is then given by

$$\rho = \frac{\sigma_{XY}}{\sigma_X\sigma_Y} = \frac{0}{\frac{1}{2}\times\frac{1}{2}} = 0.$$

3 Correlation coefficient: Equality example

Suppose now that $X = Y$. The covariance is then

$$\sigma_{XY} = E[XY] - \mu_X \mu_Y = E[X^2] - (\mu_X)^2 = \sigma_X^2.$$

The correlation coefficient is then

$$\rho = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} = \frac{\sigma_X^2}{\sigma_X \sigma_X} = 1.$$

4 Correlation coefficient: Opposite example

Suppose now that that $Y = -X + 2\mu_X$. Then

$$\mu_Y = E[Y] = E[-X + 2\mu_X] = -E[X] + 2\mu_X = \mu_X.$$

The covariance is then

$$\begin{aligned}\sigma_{XY} &= E[(X - \mu_X)(Y - \mu_Y)] \\ &= E[(X - \mu_X)((-X + 2\,\mu_X) - \mu_X)] \\ &= -E[(X - \mu_X)^2] = -\sigma_X^2.\end{aligned}$$

The correlation coefficient is then given by

$$\rho = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} = \frac{-\sigma_X^2}{\sigma_X \sigma_X} = -1.$$

5 Correlation coefficient: Intuition

The intuition here is that ρ measures how closely related X and Y are:

- When X and Y are independent, we have $\rho = 0$;
- When X and Y are equal (i.e., as related as one can be), we have $\rho = 1$.
- When X and Y are unequal as one can be, we have $\rho = -1$.

6 Correlation coefficient: Independence theorem

Theorem 1. *If X and Y are independent random variables, then, for any function $u(\cdot)$ and $v(\cdot)$,*

$$E[u(X)v(Y)] = E[u(X)]E[v(Y)].$$

In particular, if X and Y are independent random variables, then

$$\sigma_{XY} = E[XY] - \mu_X\mu_Y = E[X]E[Y] - \mu_X\mu_Y = 0.$$

Note: It is not true that $\sigma_{XY} = 0$ implies X and Y are independent! Check the textbook for an example.