Math 170E

Lecture Notes Section 4.1 [*,†]

Bivariate distributions of discrete type

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NOTE: Materials that appear in the textbook but do not appear in the lecture notes might still be tested. Please send me an email if you find typos.

[*] Version date: Monday 8th February, 2021, 14:58.

[†] This notes is based on March Boedihardjo's and Jamie Haddock's notes from the past quarters, and I would like to thank them for their generosity. "*Nanos gigantum humeris insidentes* (I am but a dwarf standing on the shoulders of giants)".

1 Bivariate random variables

So far we have discussed random variables **one at a time**.

We now increase the difficulty and discuss **two random variables** X and Y together.

2 Bivariate RVs: Discrete type

Let X and Y be two discrete random variables.

The **joint probability mass function** of X and Y is the function $f : S_X \times S_Y :\to [0, 1]$

$$f(x, y) \quad = \quad P(X = x, Y = y) \qquad \text{for all } x \in S_x, y \in S_y.$$

The **joint outcome space** S is

$$S \quad = \quad S_x \times S_y.$$

3 Bivariate RVs: Example

You first toss a fair coin, then you toss a fair dice.

Let X be the outcome of the coin toss, and Y the outcome of the dice.

The outcome space of X and Y are

$$S_X = \{0, 1\}; \qquad S_Y = \{1, 2, 3, 4, 5, 6\}.$$

The joint outcome space is

$$\begin{aligned} S =& \{0, 1\} \times \{1, 2, 3, 4, 5, 6\} \\ =& \{(i, j) \ | \ 0 \le i \le 1, \ 1 \le j \le 6\}. \end{aligned}$$

The joint pmf is given by

$$f(i, j) \quad = \quad \frac{1}{2} \times \frac{1}{6} \quad = \quad \frac{1}{12}.$$

4 Joint pmfs: Properties

Let X and Y be two discrete random variables.

The joint pmf f satisfies

- $0 \leq f(x, y) \leq 1$;
- $\sum_{(x,y)\in S} f(x, y) = 1$;
- For every subset A of S:

$$P[(X, Y) \in A] \quad = \quad \sum_{(x,y)\in A} f(x, y).$$

5 Joint pmfs and marginal pmfs

Let X and Y be discrete RVs with joint pmf f.

Then the probability mass function f_X for X is equal to

$$f_X(x) \quad = \quad P(X = x) \quad = \quad \sum_{y \in S_Y} f(x, y),$$

and is also called the **marginal probability mass function** of X.

Similarly, the probability mass function f_Y for Y is

$$f_Y(y) \quad = \quad P(Y = y) \quad = \quad \sum_{x \in S_x} f(x, y),$$

and is also called the **marginal probability mass function** of Y.

Note: For the first sum, x is fixed while y varies.

For the second sum, x varies while y is fixed.

6 Marginal pmfs: Example

Let X and Y be the coin-and-dice RVs.

We already know that $f_X(0) = f_X(1) = \frac{1}{2}$, and this can also be checked by

$$f_X(0) = \sum_{y \in S_y} f(0, y)$$

$$= f(0,1) + f(0,2) + \ldots + f(0,6) = 6 \times \frac{1}{12} = \frac{1}{2}.$$

Check $f_X(1) = \frac{1}{2}$ by the same method as an exercise.

We also already know that $f_Y(1) = \ldots = f_Y(6) = \frac{1}{6}$, and this can also be checked by

$$\begin{aligned} f_Y(1) &= \sum_{x \in S_x} f(x, 1) = f(0,1) + f(1,1) \\ &= \frac{1}{12} + \frac{1}{12} = \frac{1}{6}. \end{aligned}$$

7 Independent RVs

Two discrete random variables X and Y are **independent** if, for all $x \in S_X$, $y \in S_y$,

$$P(X = x, Y = y) \quad = \quad P(X = x)P(Y = y).$$

For example, the X and Y from the coin-and-dice example are independent.

8 Dependent RVs: Example

Alice throws a fair coin, with X being the outcome of her coin toss.

Bob is a “tsundere” so he makes his outcome of coin toss Y to always be the opposite of Alice’s coin, i,.e.,

If $X = 0$ then $Y = 1$, and if $X = 1$ then $Y = 0$.

Then X and Y are dependent, because

$$\begin{aligned} P(X = 0, Y = 0) &= 0, \qquad \text{but} \\ f_X(0)\, f_Y(0) &= \frac{1}{2} \times \frac{1}{2} \quad = \quad \frac{1}{4}. \end{aligned}$$

9 Expectations of bivariate RVs

Let $u : S_X \times S_Y \to (-\infty, \infty)$ be a function of two variables.

The **expected value** of $u(X, Y)$ is

$$E[u(X,Y)] \quad = \quad \sum_{(x,y)\in S} u(x,y)\, f(x,y).$$

10 Expectations: Example

Let X and Y be as in Alice-Bob example.

Let $u(\cdot,\cdot)$ be the function given by $u(x,y) = x + y$.

Then the expectation of $u(X,Y)$ is

$$
\begin{aligned}
&E[u(X,Y)] \\
&= u(0,0)f(0,0) + u(0,1)f(0,1) + u(1,0)f(1,0) + u(1,1)f(1,1) \\
&= \quad (0+0)\,0 \; + \; (0+1)\,\frac{1}{2} \; + \; (1+0)\,\frac{1}{2} \; + \; (1+1)\,0 \\
&= \quad 1.
\end{aligned}
$$

11 Marginal expectations

To compute the expectation of X, we can use **either** the joint pmf $f(\cdot, \cdot)$ **or** the marginal pmf f_X.

That is, μ_X is equal to

$$\mu_X \quad = \quad E[X] \quad = \quad \sum_{x \in S_X} x f_X(x),$$

and is also equal to

$$\mu_X \quad = \quad E[u(X, Y)] \quad = \quad \sum_{(x,y) \in S} x\, f(x, y),$$

where $u(x, y) = x$.

Similar formulas apply to the expection of Y.

12 Marginal variance

The variance σ_X^2 is equal to

$$\sigma_X^2 \quad = \quad \sum_{x \in S_X} (x - \mu_X)^2 f_X(x) \quad = \quad \left(\sum_{x \in S_X} x^2 f_X(x) \right) - \mu_X^2 \,,$$

and is also equal to

$$\begin{aligned} \sigma_X^2 \quad &= \quad \sum_{(x,y) \in S} (x - \mu_X)^2 \, f(x,y) \\ &= \quad \left(\sum_{(x,y) \in S} x^2 f(x,y) \right) \; - \; \mu_X^2 \,. \end{aligned}$$

13 Marginal expectations: Example

Let X and Y be as in Alice-Bob example.

We already know $\mu_X = \frac{1}{2}$ and $\sigma_X^2 = \frac{1}{4}$, since X is fair coin toss.

We can also use joint pmf to compute μ_X:

$$
\begin{aligned}
\mu_X &= \sum_{(x,y)\in S} x\, f(x,y) \\
&= 0\, f(0,0) + 0\, f(0,1) + 1\, f(1,0) + 1\, f(1,1) \\
&= 0\,(0) + 0\left(\frac{1}{2}\right) + 1\left(\frac{1}{2}\right) + 1\,(0) \quad = \quad \frac{1}{2}.
\end{aligned}
$$

We can also use joint pmf to compute σ_X^2:

$$\begin{aligned}
\sigma_X^2 &= \sum_{(x,y)\in S} (x-\mu_X)^2 f(x,y) \\
&= \left(0-\frac{1}{2}\right)^2 f(0,0) + \left(0-\frac{1}{2}\right)^2 f(0,1) \\
&\quad + \left(1-\frac{1}{2}\right)^2 f(1,0) + \left(1-\frac{1}{2}\right)^2 f(1,1) \\
&= \frac{1}{4}(0) + \frac{1}{4}\left(\frac{1}{2}\right) + \frac{1}{4}\left(\frac{1}{2}\right) + \frac{1}{4}(0) \\
&= \frac{1}{4}.
\end{aligned}$$

14 Binomial RVs: Recap

Recall the binomial RVs from Section 2.4.

- You have n independent experiments, each outcome is either a "**success**" with probability p_X or "**failure**" with probability $1 - p_X$.
- The binomial random variable X is the number of successful experiments.

We now change the experiment so that the outcome can be either "**success**", "**partial success**", or "**failure**".

15 Trinomial RVs: Intuition

- You have n independent experiments, each of them will be a "**success**" with probability p_X, "**half-success**" with probability p_Y, and "**failure**" with probability $1 - p_X - p_Y$;
- X is the number of successful experiments;
- Y is the number of half-successful experiements.

16 Trinomial RVs: Definition

The **trinomial bivariate random variable** X and Y with parameter n, p_X and p_Y has support

$$
\begin{aligned}
S &= \{(x,y) \mid 0 \le x, y \le n,\ x+y \le n\} \\
&= \{(0,0), \ldots (0,n),\ (1,0), \ldots (1,n-1),\ (2,0), \ldots,\ (n,0)\},
\end{aligned}
$$

and pmf

$$
\begin{aligned}
f(x,y) &= P(X=x, Y=y) \\
&= \frac{n!}{x!\,y!\,(n-x-y)!}\, p_X^x\, p_Y^y\, (1-p_X-p_Y)^{n-x-y},
\end{aligned}
$$

for all $x \in S_X, y \in S_Y$.

17 Trinomial RVs: Example

In a class of 40 students, there are three types of outcome for the midterm:

- Very well (80-100), probability 0.3;
- Reasonably well (50-80), probability 0.5;
- Badly (0-50), probability 0.2.

Compute the probability that

(a) 4 students are doing very well, 10 students are doing reasonably well, 26 students are doing badly.

(b) the number of students that are doing very well is exactly 11.

18 Trinomial RVs: Answer

Let X be the number of students that are doing very well.

Let Y be the number of students are doing reasonably well.

Then X and Y are trinomial random variables with $n = 40$, $p_X = 0.3$, $p_Y = 0.5$.

The question is then asking for

$$P(X = 4, Y = 10), \quad \text{and} \quad P(X = 11).$$

For the first part,

$$\begin{aligned} P(X = 4, Y = 10) &= \frac{40!}{4!\,10!\,26!}(0.3)^4\,(0.5)^{10}(0.2)^{26} \\ &= 1.233 \times 10^{-10}. \end{aligned}$$

The answer for the second part is

$$
\begin{aligned}
P(X = 11) &= \sum_{y=0}^{29} P[X = 11, Y = y] \\
&= \sum_{y=0}^{29} \frac{40!}{11!\, y!\, (29-y)!} (0.3)^{11}\, (0.5)^{y} (0.2)^{29-y}.
\end{aligned}
$$

This is a huge sum and will take you a while to compute.

Instead, note that X is the binomial random variable with

$n = 40$ and success probability 0.3.

Hence a shorter answer will be

$$
P(X = 11) = \binom{40}{11} (0.3)^{11} (0.7)^{29} = 2.074 \times 10^{-10}.
$$

19 Trinomial and binomial RVs: Connection

Theorem 1. *Let X and Y be **trinomial** RVs with parameter n, p_X, and p_Y.*
*Then X is the **binomial** RV with parameter n and p_X.*
*Similarly Y is the **binomial RV** with parameter n and p_Y.*

Note: X and Y are dependent RVs! Check the textbook to see the proof.

20 Hypergeometric RVs: recap

Recall the hypergeometric random variable X from Section 2.5:

- You have N_1 blue marbles and N_2 red marbles in an urn;
- Pick n marbles without replacement from the urn;
- The hypergeometric random variable X is the number of blue marbles that you pick.

The probability mass function f_X for X is given by

$$f_X(x) \quad = \quad P(X = x) = \frac{\binom{N_1}{x}\binom{N_2}{n-x}}{\binom{N_1+N_2}{n}}.$$

21 Bivariate hypergeometric RVs: Intuition

- You have N_1 blue marbles and N_2 red marbles, and N_3 green marbles in an urn;
- Pick n marbles without replacement from the urn;
- X is the number of blue marbles, and Y is the number of red marbles that you pick.

22 Bivariate hypergeometric RVs: Definition

The **bivariate hypergeometric** RVs X and Y with parameter N_1, N_2, N_3, n has support

$$S = \{(x, y) \mid 0 \le x \le N_1, 0 \le y \le N_2,\ x + y \le n\},$$

and pmf

$$f(x, y) = \frac{\binom{N_1}{x}\binom{N_2}{y}\binom{N_3}{n-x-y}}{\binom{N_1+N_2+N_3}{n}}.$$

Note: X is the hypergeometric RV with parameter N_1, $N_2 + N_3$, n. Similarly, Y is the hypergeometric RV with parameter N_2, $N_1 + N_3$, n.

Note: X and Y are dependent RVs! Check the textbook for proofs.