Math 170E

Lecture Notes Section 3.3 [*,†]

Normal RVs

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NOTE: Materials that appear in the textbook but do not appear in the lecture notes might still be tested. Please send me an email if you find typos.

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[†] This notes is based on March Boedihardjo's and Jamie Haddock's notes from the past quarters, and I would like to thank them for their generosity. "*Nanos gigantum humeris insidentes* (I am but a dwarf standing on the shoulders of giants)".

1 Normal RVs: Motivation

Many RVs in real life have "bell-shaped" probability density function.

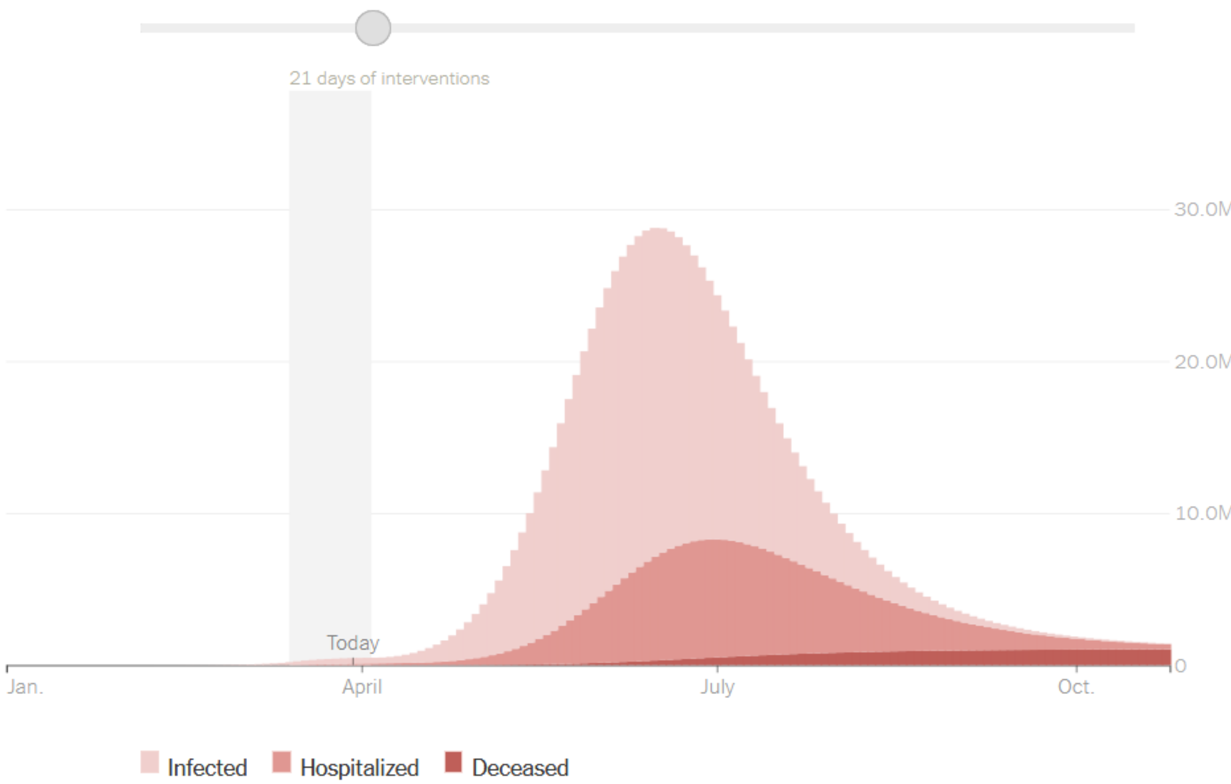


Figure 1: Predicted number of patients infected, hospitalized, and died from Covid-19, published on April 2020 by New York Times.

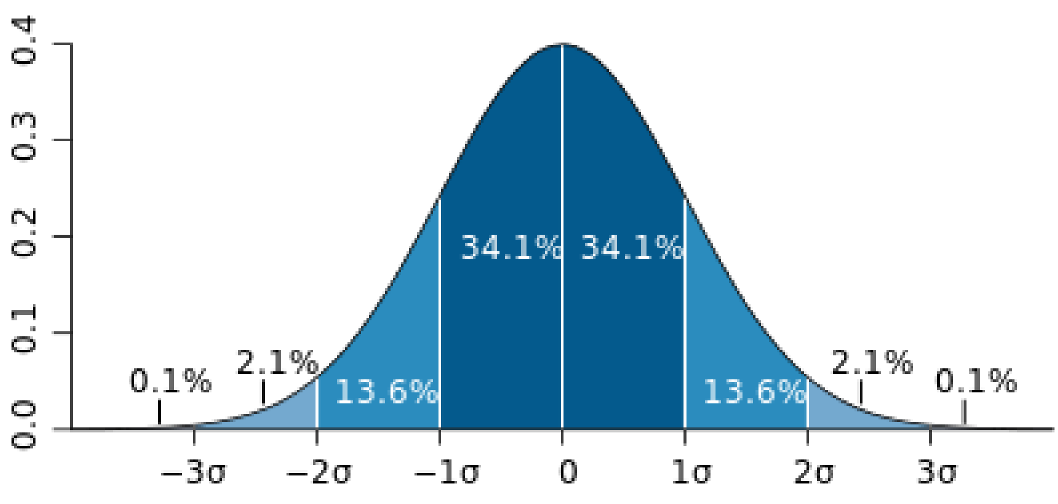
Intuitively, one should expect to see this shape whenever the process is **a sum of independent RVs**.

2 Standard normal RV: Definition

The **standard normal random variable** X is the continuous RV given by

$$\begin{aligned} S &= (-\infty, \infty); \\ f(z) &= \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-z^2}{2}\right). \end{aligned}$$

The standard normal RV is often denoted by $N(0, 1)$.



3 Standard normal RV: cdf

The cumulative distribution function of Z is denoted by

$$\Phi(z) \quad := \quad P(Z \leq z) \quad = \quad \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp^{\frac{-x^2}{2}} \, dx.$$

It is actually not possible to evaluate this integral using antiderivatives, as there is no elementary formula for it. Instead, we will use numerical approximations from Table Va and Vb of Appendix B of your textbook.

Warning: To maintain internal consistency, graders will use **Table Va and Vb** when calculating $\Phi(z)$. If your answer (**by calculator or otherwise**) differ too much from the official answer, the grader has the right to deduct points appropriately.

4 Standard normal RV: Example 1

Let Z be a standard normal RV. Compute

- $P(Z \leq 1)$;
- $P(-1 \leq Z \leq 1)$.

Answer: We have from Table Va that

$$
\begin{aligned}
P(Z \leq 1) &= \Phi(1) \quad = \quad 0.8413; \\
P(-1 \leq Z \leq 1) &= \Phi(1) - \Phi(-1) \\
&= \Phi(1) - (1 - \Phi(1)) = 2\,\Phi(1) - 1 \\
&= 2\,(0.8413) - 1 \\
&= 0.6826.
\end{aligned}
$$

5 Standard normal RV: Example 2

Let $Z = N(0,1)$. Find the constant a and b so that

$$P(Z \leq a) = 0.9808; \qquad P(Z \geq b) = 0.0024.$$

Answer: From Table Va, we get that

$$a = 2.07.$$

From Table Vb, we get that

$$b = 2.82.$$

6 Standard normal RV: mean, variance, mgf

Theorem 1. *Let Z be the standard normal random variable $N(0,1)$. The moment generating function of Z is given by*

$$M(t) = \exp\left(\frac{t^2}{2}\right).$$

The mean and variance are given by

$$\mu = 0; \qquad \sigma^2 = 1.$$

Please check the textbook for proofs.

7 General normal RVs

The **normal random variable** X with mean μ and variance σ^2 is the continuous random variable given by

$$
\begin{aligned}
S &= (-\infty, \infty); \\
f(x) &= \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right].
\end{aligned}
$$

This random variables is often denoted by $N(\mu, \sigma^2)$.

8 General normal RVs: mean, variance, mgf

Theorem 2. *Let X be $N(\mu, \sigma^2)$. The moment generating function of X is given by*

$$M(t) = \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right).$$

The mean and variance of X are given by μ and σ^2 respectively.

9 General normal RVs: cdf

Computing probabilities for $X = N(\mu, \sigma^2)$ directly is not recommended.

Instead, we usually convert it to probabilities for standard normal distribution, then compute the latter.

Theorem 3. *Let X be $N(\mu, \sigma^2)$, and let Z be*

$$Z := \frac{X - \mu}{\sigma}.$$

Then Z is $N(0,1)$.

In particular, for all x:

$$\begin{aligned} P(X \le x) &= P\left(Z \le \frac{x-\mu}{\sigma}\right) &= \Phi\left(\frac{x-\mu}{\sigma}\right); \\ P(X \ge x) &= P\left(Z \ge \frac{x-\mu}{\sigma}\right) &= 1 - \Phi\left(\frac{x-\mu}{\sigma}\right). \end{aligned}$$

10 General normal RVs: Example

Let $X = N(1, 0.25)$. Compute $P(-0.5 \leq X \leq 1.5)$.

Answer: We have

$$\begin{aligned} P(X \leq -0.5) &= \Phi\left(\frac{(-0.5)-1}{0.5}\right) = \Phi(-3) \\ P(X \leq 1.5) &= \Phi\left(\frac{(1.5)-1}{0.5}\right) = \Phi(1). \end{aligned}$$

Hence

$$\begin{aligned} P(-0.5 \leq X \leq 1.5) &= P(X \leq 1.5) - P(X \leq -0.5) \\ &= \Phi(1) - \Phi(-3) = \Phi(1) - (1 - \Phi(3)) \\ &= \Phi(1) + \Phi(3) - 1 = 0.8413 + 0.9987 - 1 \\ &= 0.84. \end{aligned}$$

11 Normal RVs and Chi-square Rvs: Relationships

Theorem 4. *Let* $X = N(\mu, \sigma^2)$. *Then the random variable* $V = \frac{(X-\mu)^2}{\sigma^2}$ *has chi-square* $\chi^2(1)$ *distribution.*

12 Normal RVs and Chi-square Rvs: Example

Let $Z = N(0, 1)$. Find a such that

$$P(|Z| < a) = 0.1.$$

Answer: Note that

$$P(|Z| < a) = P(Z^2 < a^2).$$

So we need to find a such that

$$P(Z^2 < a^2) = 0.1.$$

By Table IV in Appendix B for $\chi^2(1)$, we have

$$a^2 = 0.016; \qquad \text{and therefore} \qquad a = 0.126.$$