Math 170E

Lecture Notes Section 3.2 *†

Exponential, Gamma, and Chi-Square RVs

Instructor: Swee Hong Chan

NOTE: Materials that appear in the textbook but do not appear in the lecture notes might still be tested. Please send me an email if you find typos.

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†This notes is based on March Boedihardjo's and Jamie Haddock's notes from the past quarters, and I would like to thank them for their generosity. "*Nanos gigantum humeris insidentes* (I am but a dwarf standing on the shoulders of giants)".

1 Exponential RVs

The **exponential random variable** X with **parameter** (mean) θ is the continuous RV given by

$$\begin{aligned} S &= [0, \infty), \\ f(x) &= \frac{1}{\theta} e^{-x/\theta} \quad \text{for} \quad 0 \leq x < \infty. \end{aligned}$$

2 Exponential RVs: Example

Suppose that the lifetime of an iPhone has an exponential distribution with a mean life of 3 years.

Let X denote the life of the iPhone of the friendly instructor's (FI) sister.

- What is the probability that the iPhone is broken one year after it is purchased?
- What is the probability that the iPhone lasts longer than 3 years?

3 Exponential RVs: Answer

Here X is an exponential RV with $\theta = 3$, and the question is asking for

$$P(X \leq 1) \qquad \text{and} \qquad P(X \geq 3).$$

We then have

$$\begin{aligned} P(X \leq 1) &= \int_0^1 \frac{e^{-x/3}}{3}\,dx = \left[-e^{-x/3}\right]_0^1 \\ &= -e^{-1/3} - (-e^{-0/3}) = 1 - e^{-1/3}. \end{aligned}$$

$$\begin{aligned} P(X \geq 3) &= \int_3^\infty \frac{e^{-x/3}}{3}\,dx = \left[-e^{-x/3}\right]_3^\infty \\ &= -e^{-\infty/3} - (-e^{-3/3}) = e^{-1}. \end{aligned}$$

4 Exponential and Poisson RVs: Relationships

Let X_1 be the time you wait until the first random event occurs.

Let X_2 the time in between the first random event and the second random event.

Let X_3 the time in between the second random event and the third random event, ...

Let Y be the number of random events that occur in the time interval of one unit.

Theorem 1. *Suppose that X_1, X_2,. . . are* ***independent exponential RVs*** *with mean θ.*

Then Y is the ***Poisson RV*** *with mean $\frac{1}{\theta}$.*

5 Exponential and Poisson RVs: Example

Continuing from the previous example, suppose that FI's sister will immediately replace the iPhone with the exact same model whenever the current phone is broken.

Let Y be the number of phones FI's sister has replaced in 1 year.

- What is the probability that $Y = 0$?
- What is the probability that $Y = 2$?
- What is the probability that $Y = 5$?

6 Exponential and Poisson RVs: Answer

For $Y = 0$, the first iPhone needs to last longer than 1 year (i.e., $X \geq 1$). So we have

$$
\begin{aligned}
P(Y = 0) &= P(X \geq 1) = 1 - P(X < 1) \\
&= 1 - (1 - e^{-1/3}) = e^{-1/3}.
\end{aligned}
$$

We can also use the theorem that Y is the Poisson RV with mean $\frac{1}{3}$:

$$
P(Y = 0) = \left(\frac{1}{3}\right)^0 \frac{e^{-1/3}}{0!} = e^{-1/3}.
$$

We can do a direct calculation to compute $P(Y = 2)$, but it will be really complicated.

So we will just use the fact that Y is the Poisson random variable with rate $\frac{1}{3}$. Recall that Y has pmf

$$P(Y = x) \quad = \quad e^{-1/3}\frac{(1/3)^x}{x!} \quad \text{for} \quad x \in \{0, 1, 2, 3, \ldots\}.$$

Hence we have

$$\begin{aligned} P(Y = 2) \quad &= \quad e^{-1/3}\frac{(1/3)^2}{2!} \quad = \quad \frac{e^{-1/3}}{18}; \\ P(Y = 5) \quad &= \quad e^{-1/3}\frac{(1/3)^5}{5!}. \end{aligned}$$

7 Exponential and Poisson RVs: Reverse relationships

Let Y be the number of occurrences of a random event in a unit time interval.

Let X_1 be the time you wait until the first random event occurs.

Let X_2 the time in between the first random event and the second random event.

Let X_3 the time in between the second random event and the third random event, $\ldots$.

Theorem 2. *Suppose that Y is the* ***Poisson RV*** *with mean λ. Then X_1, X_2,... are* ***independent exponential RVs*** *with mean $\frac{1}{\lambda}$.*

8 Exponential RVs: mean, variance, mgf

Theorem 3. *Let X be the exponential random variable with mean θ. The moment generating function of X is*

$$M(t) = \frac{1}{1-\theta t} \qquad \text{for } t < \frac{1}{\theta}.$$

The mean and variance of X is

$$\mu = \theta; \qquad\qquad \sigma^2 = \theta^2.$$

Please check the textbook for proofs.

9 Gamma RVs: Mathematical definition

The **gamma random variable** X with parameter α and θ is the continuous RV given by

$$\begin{aligned} S &= [0, \infty); \\ f(x) &= \frac{1}{\Gamma(\alpha)} \frac{x^{\alpha-1}}{\theta^{\alpha}} \exp\left[-\frac{x}{\theta}\right]. \end{aligned}$$

Here $\Gamma(\alpha)$ is the **gamma function**, which is equal to $(\alpha-1)!$ if α is a positive integer. For general positive real number α, the function is

$$\Gamma(\alpha) \quad := \quad \int_0^\infty y^{\alpha-1} e^{-y} \, dy.$$

10 Gamma RVs and Exponential RVs: Relationships

Let X_1 be the time you wait until the first random event occurs.

Let X_2 the time in between the first random event and the second random event, ...

Let X_3 the time in between the second random event and the third random event, ...

Let Z_α (where α is a positive integer) is the total waiting time until the α-th random event occurs,

$$Z_\alpha \quad = \quad X_1 + X_2 + \ldots + X_\alpha .$$

Theorem 4. *Suppose that X_1, X_2,... are* ***independent exponential RVs*** *with mean θ. Then Z_α is the* ***Gamma RV*** *with parameter α and θ.*

11 Gamma RVs: Example

Recall from the previous example that the lifetime of an iPhone is an exponential RV with mean 3 years, and FI's sister replaces his current iPhone with a new one whenever the current phone is broken.

Compute the probability that FI's sister will replace her phones at least twice within 6 years.

12 Gamma RVs: Answer

Let Z be the time the FI's sister spent using the first two phones. This is the gamma RV with $\alpha = 2$ and $\theta = 3$, and the question is asking for $P(Z \leq 6)$.

Then

$$\begin{aligned} P(Z \leq 6) &= \int_0^6 \frac{1}{(2-1)!} \frac{x^{2-1}}{3^2} e^{-\frac{x}{3}} \, dx \\ &= \frac{1}{9} \int_0^6 x \, e^{-\frac{x}{3}} \, dx. \end{aligned}$$

Let's use integration by parts. Recall the formula

$$\int f \, g' \, dx = f \, g - \int f' \, g \, dx.$$

We will apply this formula with

$$f = x; \qquad g' = e^{-\frac{x}{3}}.$$

This gives us

$$
\begin{aligned}
\int x\, e^{-\frac{x}{3}}\, dx &= x\,((-3)e^{\frac{-x}{3}}) - \int (1)\,((-3)e^{\frac{-x}{3}})\, dx \\
&= -\,3\,x\,e^{\frac{-x}{3}} + 3\int e^{\frac{-x}{3}}\, dx \\
&= -\,3\,x\,e^{\frac{-x}{3}} - 9\ e^{\frac{-x}{3}}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
P(Z \le 6) &= \frac{1}{9}\int_0^6 x\, e^{-\frac{x}{3}}\, dx \quad = \quad \frac{1}{9}\left[-3\,x\,e^{\frac{-x}{3}} - 9\ e^{\frac{-x}{3}}\right]_0^6 \\
&= \frac{1}{9}\left(-3\,(6)\,e^{\frac{-6}{3}} - 9\ e^{\frac{-6}{3}}\right) - \frac{1}{9}\left(-3\,(0)\,e^{\frac{-0}{3}} - 9\ e^{\frac{-0}{3}}\right) \\
&= \frac{-27e^{-2}}{9} - \frac{-9}{9} \quad = \quad 1 - 3e^{-2}.
\end{aligned}
$$

This question can also be solved by using Poisson random variable interpretaion. Please do it as an exercise.

13 Gamma RVs: mean, variance, mgf

Theorem 5. *Let X be a gamma random variable with parameter α and θ. The moment generating function of X is*

$$M(t) = \frac{1}{(1-\theta\, t)^{\alpha}}, \qquad \text{for } t < \frac{1}{\theta}.$$

The mean and variance of X are given by

$$\mu = \alpha\,\theta; \qquad \sigma^2 = \alpha\,\theta^2.$$

Check the textbook for proofs.

14 Chi-square RVs

The **chi-square random variable** X with r **degrees of freedom** is the continuous random variable given by

$$\begin{aligned} S &= [0, \infty); \\ f(x) &= \frac{1}{\Gamma(\frac{r}{2})} \frac{x^{\frac{r}{2}-1}}{2^{\frac{r}{2}}} e^{-\frac{x}{2}}. \end{aligned}$$

Note that chi-square random variable is the gamma random variable with $\theta = 2$ and $\alpha = \frac{r}{2}$, and is denoted by $\chi^2(r)$.

This RV will be featured again in Math 170S, and in particular $\chi^2(1)$ will feature prominently when we study normal RVs in the next section.

15 Chi-square RVs: Example

Let X be the chi-square RV with $r = 8$. Compute the probability

$$P(3.490 \le X \le 17.54).$$

Answer: Solving this question directly using integrals is time-consuming.

Instead, let's use the numerical approximation in Table IV in Appendix B. Then

$$\begin{aligned} P(3.490 \le X \le 17.54) &= P(X \le 17.54) - P(X \le 3.490) \\ &= 0.975 - 0.100 = 0.875. \end{aligned}$$

$\square$