Math 170E

Lecture Notes Section 2.6 [*†]

Negative binomial distribution

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NOTE: Materials that appear in the textbook but do not appear in the lecture notes might still be tested. Please send me an email if you find typos.

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†This notes is based on March Boedihardjo's and Jamie Haddock's notes from the past quarters, and I would like to thank them for their generosity. "*Nanos gigantum humeris insidentes* (I am but a dwarf standing on the shoulders of giants)".

1 Negative binomial distribution

Your friendly instructor is playing a trading card game where he tried to gather three copies of a rare card called "Blue Eyes White Dragon".

When buying a random card, there is probability p that this card is a Blue Eyes, and your instructor intends to continue buying cards until he gets exactly 3 copies of Blue Eyes.

Let X be the number of cards that your instructor ended up buying. (**Note**: He needs to buy at least 3 card.)

The probability that $X = 3$ is

$$\begin{aligned} P(X = 3) &= P(\text{All three cards are Blue Eyes}) \\ &= p^3. \end{aligned}$$

For $X = 4$, the following needs to happen:

- The fourth card must be Blue Eyes;
- The first three cards has exactly two blue eyes.

So, the probability that $X = 4$ is

$$P(X = 4) \quad = \quad p \times \binom{3}{2} p^2 (1-p) \quad = \quad \binom{3}{2} p^3 (1-p).$$

For $X = 5$, the following needs to happen:

- The fifth card must be Blue Eyes;
- The first four cards has exactly two blue eyes.

So, the probability that $X = 5$ is

$$P(X = 5) \quad = \quad p \times \binom{4}{2} p^2 (1-p)^2 \quad = \quad \binom{4}{2} p^3 (1-p)^2.$$

For $X = x$, the following needs to happen:

- The x-th card must be Blue Eyes;
- The first $x - 1$ cards has exactly two blue eyes.

So, the probability that $X = x$ is

$$\begin{aligned} P(X = x) &= p \times \binom{x-1}{2} p^2 (1-p)^{x-3} \\ &= \binom{x-1}{2} p^3 (1-p)^{x-3}. \end{aligned}$$

2 Negative binomial distribution: Intuitive definition

- One is performing successive Bernoulli trials until one gets r many successes;
- The success probability of each Bernoulli trial is p;
- **Negative binomial** random variable X is the number of trials it takes **to get r many successes**.

This is not to be confused with **binomial** random variable Y, which is **the number of successes** in n trials.

3 Negative binomial distribution: Mathematical definition

The **Negative binomial random variable** X with **parameter** p and r has support and pmf

$$
\begin{aligned}
S &= \{r, r+1, r+2, \ldots\}; \\
f(x) &= \binom{x-1}{r-1} p^r (1-p)^{x-r} \qquad x \in \{r, r+1, r+2, \ldots\}.
\end{aligned}
$$

4 Negative binomial distribution: mgf, mean, and variance

Theorem 1. *Let X be the negative binomial random variable with parameter p and r. The moment generating function of X is*

$$M(t) = \frac{(pe^t)^r}{[1-(1-p)e^t]^r}.$$

The mean and the variance of X is

$$\mu = \frac{r}{p} \qquad \textit{and} \qquad \sigma^2 = \frac{r(1-p)}{p^2}.$$

Please check the textbook for the proof of the theorem above.

5 Geometric random variable

Recall geometric random variable from Section 2.3.

Geometric random variable is the negative binomial RV when $\mathbf{r = 1}$ (i.e., keep playing until 1 success occurs).

The support and pmf is

$$\begin{aligned} S &= \{1, 2, 3, 4, \ldots\}; \\ f(x) &= (1-p)^{x-1}p \qquad \text{for } x \in \{1, 2, 3, 4, \ldots\}. \end{aligned}$$

6 Geometric random variable: Theorem

Theorem 2. *Let X be the geometric RV with success probability p. Then, for any nonnegative integer k,*

$$
\begin{aligned}
P(X > k) &= (1-p)^k; \\
P(X \leq k) &= 1-(1-p)^k.
\end{aligned}
$$

7 Geometric random variable: Proof

Here is a proof of the theorem: (BT)

$$
\begin{aligned}
& P(X > k) \\
= \quad & P(X = k+1) \ + \ P(X = k+2) \ + \ P(X = k+3) \ + \ \ldots \\
= \quad & (1-p)^k p \ + \ (1-p)^{k+1} p \ + \ (1-p)^{k+2} p \ + \ \ldots \\
= \quad & (1-p)^k p \ \big(1 \ + \ (1-p) \ + \ (1-p)^2 \ + \ \ldots\big) \\
= \quad & (1-p)^k p \ \sum_{i=0}^{\infty} (1-p)^i.
\end{aligned}
$$

Now, recall the Taylor series

$$
\frac{1}{1-z} \quad = \quad \sum_{i=0}^{\infty} z^i \quad = \quad 1 + z + z^2 + z^3 + \ldots.
$$

Subtituting $z = 1 - p$ to the equation for $P(X > k)$,

$$\begin{aligned}
P(X > k) &= (1-p)^k p \ \sum_{i=0}^{\infty} (1-p)^i \\
&= (1-p)^k p \ \left(\frac{1}{1-(1-p)} \right) \\
&= (1-p)^k p \ \left(\frac{1}{p} \right) \\
&= (1-p)^k.
\end{aligned}$$

8 Geometric random variable: Example

A basketball player is shooting free throws, and each shot is a success with probability $p = 0.7$.

What is the probability that the number of tries until she gets one shot is at least 3?

What is the probability that she needs to shoot strictly less than 3 times to get one shot?

9 Geometric random variable: Answer

This is a geometric random variable with $p = 0.7$. The questions are asking for

$$P(X \geq 3) \quad \text{and} \quad P(X < 3).$$

Hence we have

$$\begin{aligned} P(X \geq 3) &= P(X > 2) \\ &= (1 - 0.7)^2 = (0.3)^2 = 0.09. \\ P(X < 3) &= P(X \leq 2) \\ &= 1 - (1 - 0.7)^2 = 1 - (0.3)^2 = 0.91. \end{aligned}$$