Math 170E

Lecture Notes Section 2.4 [*,†]

Bernoulli and Binomial distribution

Instructor: Swee Hong Chan

NOTE: Materials that appear in the textbook but do not appear in the lecture notes might still be tested. Please send me an email if you find typos.

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[†]This notes is based on March Boedihardjo's and Jamie Haddock's notes from the past quarters, and I would like to thank them for their generosity. "*Nanos gigantum humeris insidentes* (I am but a dwarf standing on the shoulders of giants)".

1 Bernoulli distribution: Example

A basketball player is shooting a free throw with success probability p and failure probability $q = 1 - p$.

Let X be the random variable associated to this random experiment by

$$X(\text{success}) = 1; \qquad X(\text{failure}) = 0.$$

Ths support of X is

$$\{0, 1\}.$$

The pmf of X is

$$f(0) = (1 - p); \qquad f(1) = p.$$

2 Bernoulli distribution

A **Bernoulli trial/experiment**, is a random experiment where there are only two possible outcomes, **success** or **failure**.

A random variable X has **Bernoulli distribution** with **parameter** p if

- The support of X is $\{0, 1\}$; and
- The pmf of X is

$$f(0) = (1 - p); \qquad f(1) = p.$$

3 Bernoulli distribution: Expectations

Let X be a Bernoulli random variable with parameter p.

The mean of X is

$$\mu_X = E[X] = (0) \times P(X = 0) + (1) \times P(X = 1) \;=\; p.$$

The variance of X is

$$\begin{aligned}
\sigma_X^2 &= E[X^2] - \mu_X^2 \\
&= (0)^2 \times P(X = 0) \;+\; (1)^2 P(X = 1) - \mu_X^2 \\
&= p - p^2 \quad = \quad p(1-p) \\
&= pq.
\end{aligned}$$

The standard deviation of X is

$$\sigma_X = \sqrt{pq}.$$

4 Binomial distribution: Example

Suppose that the same basketball player is now asked to perform 100 free throws.

Let Y be the number of successful free throws.

For the event that $Y = 0$,

$$P(Y = 0) \quad = \quad P(\text{all shots fail}) \quad = \quad (1-p)^{100}.$$

For the event that $Y = 1$,

$$\begin{aligned}
P(Y = 1) \quad &= \quad P(\text{only one shot is successful}) \\
= \quad & \sum_{i=1}^{100} P(\text{success only at i-th throw}) \\
= \quad & (p)(1-p)^{99} \; + \; (1-p)(p)(1-p)^{98} \; + \; \ldots \; + \; (1-p)^{99}p \\
= \quad & 100\, p(1-p)^{99}.
\end{aligned}$$

For the event that $Y = 2$,

$$
\begin{aligned}
P(Y = 2) &= \sum_{1 \leq i < j \leq 100} P(\text{success only at } i\text{-th and } j\text{-th throw}) \\
&= \sum_{1 \leq i < j \leq 100} p^2 (1-p)^{100-2}; \\
&= \binom{100}{2} p^2 (1-p)^{98}.
\end{aligned}
$$

By the same argument, for all $x \in \{0, 1, \ldots, 100\}$,

$$
P(Y = x) = \binom{100}{x} p^x (1-p)^{100-x}.
$$

5 Binomial distribution: Definition

The random variable X has the **binomial distribution** with **parameter** n and p if

- The support of X is

$$\{0, 1, 2, 3, \ldots, n\},$$

- The pmf of X is

$$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x} \qquad \text{for } x \in \{0, 1, \ldots, n\}.$$

6 Binomial distribution: Equivalent definition

The binomial random variable X with parameter n and p can also be described as follows:

- A Bernoulli experiment is performed n times;
- The trials are independent;
- The success probability of each trial is p;
- X is the number of success in n experiements.

7 Binomial distribution: mgf

Theorem 1. *Let X be the binomial random variable with parameter n and p. The mgf of X is*

$$M(t) = [(1-p)+pe^t]^n.$$

8 Binomial random variable: mgf proof

The mgf of a binomial random variable is

$$\begin{aligned}
M(t) &= \sum_{x=0}^{n} e^{tx} f(x) \\
&= \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x (1-p)^{n-x} = \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x (1-p)^{n-x}.
\end{aligned}$$

Now, recall the binomial theorem:

$$(a+b)^n = \sum_{x=0}^{n} \binom{n}{x} b^x a^{n-x}.$$

Applying the binomial theorem to the formula for $M(t)$ with $a = 1-p$ and $b = pe^t$, we get

$$M(t) = [(1-p) + pe^t]^n.$$

9 Binomial random variable: mean and variance

Theorem 2. *Let X be the binomial random variable with parameter n and p. The mean and the variance of X is*

$$\mu = np; \qquad \sigma^2 = np(1-p).$$

10 Binomial random variable: mean proof

Let Y_i be the Bernoulli random variable for the i-th experiment. Then

$$\begin{aligned}
\mu &= E[X] \\
&= E[Y_1 + Y_2 + \ldots + Y_n] \\
&= E[Y_1] + E[Y_2] + \ldots + E[Y_n] \\
&= p + p + \ldots + p \\
&= np.
\end{aligned}$$

11 Binomial random variable: variance proof

The second moment of X is equal to (BT)

$$
\begin{aligned}
E[X^2] &= E[(Y_1 + Y_2 + \ldots + Y_n)^2] \\
&= E\left[(Y_1^2 + \ldots + Y_n^2) + 2(Y_1 Y_2 + Y_1 Y_3 + \ldots + Y_{n-1} Y_n)\right] \\
&= \left(\sum_{i=1}^{n} E[Y_i^2]\right) + 2 \sum_{1 \le i < j \le n} E[Y_i Y_j] \\
&= \left(\sum_{i=1}^{n} p\right) + 2 \sum_{1 \le i < j \le n} E[Y_i]\, E[Y_j] \\
&= np \ + \ 2\binom{n}{2} p^2 \quad = \quad np \ + \ n(n-1)p^2.
\end{aligned}
$$

The variance of X is then equal to :

$$
\begin{aligned}
\sigma^2 &= E[X^2] - (E[X])^2 \quad = \quad np + n(n-1)p^2 - (np)^2 \\
&= np - np^2 \quad = \quad np(1-p).
\end{aligned}
$$

Note 1: The formula for the mean, variance, and the mgf of Bernoulli and binomial distributions are included in the table in the first page of your textbook.

Note 2: You can also use derivatives of $M(t)$ to find the mean and the variance of the binomial random variable; check the textbook. (The methods presented here are arguably easier.)