Math 170E

Lecture Notes Section 2.3 *†

Mean, variance, and moments

Instructor: Swee Hong Chan

NOTE: Materials that appear in the textbook but do not appear in the lecture notes might still be tested. Please send me an email if you find typos.

*Version date: Friday 22nd January, 2021, 12:54.

†This notes is based on March Boedihardjo's and Jamie Haddock's notes from the past quarters, and I would like to thank them for their generosity. "*Nanos gigantum humeris insidentes* (I am but a dwarf standing on the shoulders of giants)".

1 Mean and variance: Definition

Let X be a random variable.

The **mean** of X is

$$\mu \quad := \quad E[X].$$

The **variance** of X is

$$\sigma^2 \quad := \quad \mathrm{Var}(X) \quad := \quad E[(X-\mu)^2] \quad = \quad E[X^2]-\mu^2.$$

The **standard deviation** of X is

$$\sigma \quad := \quad \sqrt{\mathrm{Var}(X)}.$$

Note: Variance and standard deviation are always non-negative.

2 Mean and Variance: Example

Consider these three types of investments:

- Investment X: you always get 1 USD.
- Investment Y: Half the time you get 2 USD, the other half of the time you get nothing.
- Investment Z: 9 out of 10 times you get 10 USD, but 1 out of 10 times you lose 80 USD.

What is the **average return** of each investment?

Which investment is the **riskiest**?

3 Mean and Variance: Answer

The average return of the investment is the **mean**, so

$$\begin{aligned}
\mu_X &= E[X] = 1 \times 1 = 1; \\
\mu_Y &= E[Y] = 0 \times \frac{1}{2} + 2 \times \frac{1}{2} = 1; \\
\mu_Z &= E[Z] = 10 \times \frac{9}{10} + (-80) \times \frac{1}{10} = 1.
\end{aligned}$$

So the three investments have the same amount of average return.

In terms of risk, it seems that X is the safest investment, while Z is the riskiest investment. Let's see if we can observe that from computing the variance:

$$\begin{aligned}
\sigma_X^2 &= E[(X-\mu_X)^2] = (1-1)^2 \times 1 = 0; \\
\sigma_Y^2 &= E[(Y-\mu_Y)^2] = (0-1)^2 \times \frac{1}{2} + (2-1)^2 \times \frac{1}{2} \\
&= 1; \\
\sigma_Z^2 &= E[(Z-\mu_Z)^2] \\
&= (10-1)^2 \times \frac{9}{10} + (-80-1)^2 \times \frac{1}{10} \\
&= 729.
\end{aligned}$$

The riskier the investment is, the larger the variance is.

Thus **mean** measures the **average return**, while **variance** measures the **risk**.

4 Moment: Definition

For any positive integer r, the r-th **moment** of X is

$$E[X^r].$$

5 Moment: Example

Let X be the random variable with $S_X = \{-1, 0, 1\}$, and with pmf

$$f(-1) \;=\; f(0) \;=\; f(1) \;=\; \frac{1}{3}.$$

The first, second, third, fourth moments of X are

$$\begin{aligned}
E[X] &= (-1) \times \frac{1}{3} + (0) \times \frac{1}{3} + (1) \times \frac{1}{3} \;=\; 0; \\
E[X^2] &= (-1)^2 \times \frac{1}{3} + (0)^2 \times \frac{1}{3} + (1)^2 \times \frac{1}{3} \;=\; \frac{2}{3}; \\
E[X^3] &= (-1)^3 \times \frac{1}{3} + (0)^3 \times \frac{1}{3} + (1)^3 \times \frac{1}{3} \;=\; 0; \\
E[X^4] &= (-1)^4 \times \frac{1}{3} + (0)^4 \times \frac{1}{3} + (1)^4 \times \frac{1}{3} \;=\; \frac{2}{3}.
\end{aligned}$$

For general r, we have

- $E[X^r] = 0$ if r is odd;
- $E[X^r] = \frac{2}{3}$ if r is even.

6 Moment generating function: definition

The **moment generating function** (mgf) of X is

$$M(t) \quad := \quad E[e^{tX}] \quad = \quad \sum_{x \in S} e^{tx} f(x)$$

$$= \quad 1 + E[X]\, t + E[X^2]\, \frac{t^2}{2!} + E[X^3]\, \frac{t^3}{3!} + \ldots + E[X^r]\, \frac{t^r}{r!} + \ldots .$$

Note that t is the **unknown variable** of the moment generating function.

Note that the last equality is not obvious. Try to prove it yourself.

7 Moment generating function: example level 0

Let X be the random variable from before with pmf

$$f(-1) \ = \ f(0) \ = \ f(1) \ = \ \frac{1}{3}.$$

Its moment generating function is

$$\begin{aligned} M(t) \quad &= \quad E[e^{tX}] \\ &= \quad e^{t(-1)}P(X=-1) \ + \ e^{t(0)}P(X=0) \ + \ e^{t(1)}P(X=1) \\ &= \quad e^{-t} \times \frac{1}{3} \ + \ 1 \times \frac{1}{3} \ + \ e^{t} \times \frac{1}{3} \\ &= \quad \frac{1}{3}\left(e^{t} + e^{-t} + 1\right). \end{aligned}$$

8 Moment generating function: theorem

Theorem 1. *If two random variables have the same moment generating function, then they have the same support and pmf.*

Note: This is the main reason why mgf is important!

9 Mgf: example level 1

Suppose that X has the mgf

$$M(t) \quad = \quad e^{5t} \times \frac{3}{6} \; + \; e^{43t} \times \frac{2}{6} \; + \; e^{121t} \times \frac{1}{6},$$

Find the support and the pmf of X.

Answer: By comparing it with the formula

$$M(t) = \sum_{x \in S} e^{tx} f(x),$$

we can conclude that $S_X \; = \; \{5, 43, 121\}$, with pmf

$$f(5) = \frac{3}{6}; \qquad f(43) = \frac{2}{6}; \qquad f(121) = \frac{1}{6}.$$

$\square$

10 Mgf: example level 2

Suppose that X has the mgf

$$M(t) = \frac{e^t/2}{1 - e^t/2}.$$

Find the support and the pmf of X.

11 Mgf: answer level 2

We will change the form of $M(t)$ to something that resembles $\sum_{x \in S} e^{tx} f(x)$.

Our friend here is the following Taylor series formula:

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \dots$$

Substituting $z = e^t/2$ to the formula above, (BT)

$$\begin{aligned}
M(t) &= \frac{e^t/2}{1 - e^t/2} \\
&= \frac{e^t}{2}\left(1 + \frac{e^t}{2} + \frac{e^{2t}}{2^2} + \frac{e^{3t}}{2^3} + \dots\right) \\
&= \left(e^t\right)\left(\frac{1}{2}\right)^1 + \left(e^{2t}\right)\left(\frac{1}{2}\right)^2 + \left(e^{3t}\right)\left(\frac{1}{2}\right)^3 + \dots.
\end{aligned}$$

By comparing the formula

$$M(t) = (e^t)\left(\frac{1}{2}\right)^1 + (e^{2t})\left(\frac{1}{2}\right)^2 + (e^{3t})\left(\frac{1}{2}\right)^3 + \ldots$$

to the formula $M(t) = \sum_{x \in S} e^{tx} f(x)$, we conclude that

$$\begin{aligned} S &= \{1, 2, 3, 4, \ldots\}; \\ f(x) &= \left(\frac{1}{2}\right)^x \qquad \text{for } x \in \{1, 2, 3, 4, \ldots\}. \end{aligned}$$

This is the support and the pmf of the geometric random variable with $p = 1/2$.

12 Mgf: another theorem

Theorem 2. *Let $r \geq 1$. The r-th moment of X is equal to the r-th derivative of the mgf $M(t)$ with $t = 0$,*

$$E[X^r] = M^{(r)}(0).$$

13 Mgf: example level 3

The mgf of X is given by

$$M(t) = \frac{pe^t}{1 - qe^t},$$

where $q = 1 - p$.

Find the mean and the variance of X.

14 Mgf: answer level 3

We write the numerator and denominator of $M(t)$ as

$$a(t) = pe^t; \qquad b(t) = 1 - qe^t.$$

Note that

$$a'(t) = pe^t; \qquad b'(t) = -qe^t.$$

Taking the derivative of $M(t)$, we have

$$\begin{aligned} M'(t) &= \left(\frac{a(t)}{b(t)}\right)' \\ &= \frac{a'(t)\,b(t) \;-\; a(t)\,b'(t)}{b(t)^2} \qquad \text{(product rule)} \\ &= \frac{(pe^t)(1-qe^t) - (pe^t)(-qe^t)}{(1-qe^t)^2} \\ &= \frac{pe^t}{(1-qe^t)^2}. \end{aligned}$$

Taking the derivative of $M'(t)$, we have

$$\begin{aligned}
M''(t) &= \left(\frac{pe^t}{(1-qe^t)^2}\right)' = \left(\frac{a(t)}{b(t)^2}\right)' \\
&= \frac{a'(t)\,b(t) \;-\; 2\,a(t)\,b'(t)}{b(t)^3} \qquad \text{(product rule again)} \\
&= \frac{(pe^t)(1-qe^t) - (2)(pe^t)(-qe^t)}{(1-qe^t)^3} \\
&= \frac{pe^t(1+qe^t)}{(1-qe^t)^3}.
\end{aligned}$$

Plugging in $t = 0$ to the formula for $M'(t)$ and $M''(t)$, we get

$$\begin{aligned}
M'(0) &= \frac{pe^0}{(1-qe^0)^2} &= \frac{p}{(1-q)^2} &= \frac{1}{p}; \\
M''(0) &= \frac{pe^0(1+qe^0)}{(1-qe^0)^3} &= \frac{p(1+q)}{(1-q)^3} &= \frac{1+q}{p^2}.
\end{aligned}$$

Finally, we have

$$\begin{aligned}
\mu_X &= E[X] &= M'(0) &= \frac{1}{p}; \\
\sigma_X^2 &= E[X^2] - \mu_X^2 &= M''(0) - (M'(0))^2 & \\
&= \frac{1+q}{p^2} - \frac{1}{p^2} &= \frac{q}{p^2}. &
\end{aligned}$$