Math 170E

Lecture Notes Section 2.2 *†

Mathematical expectations

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NOTE: Materials that appear in the textbook but do not appear in the lecture notes might still be tested. Please send me an email if you find typos.

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†This notes is based on March Boedihardjo's and Jamie Haddock's notes from the past quarters, and I would like to thank them for their generosity. "*Nanos gigantum humeris insidentes* (I am but a dwarf standing on the shoulders of giants)".

1 Expectation: Motivation

Suppose you are a casino owner, and you pay the player x dollars everytime it roll an x on a 6-sided die.

How much do you need to charge for each game so that you break even **in the long run**?

2 Expectation: Definition

Let u be any function. The **expectation/expected value** of u for a random variable X of discrete type is

$$\sum_{x \in S} u(x)\, P(X = x).$$

We denote the expectation of u by $E[u(X)]$.

NOTE: The summation here can be an infinite sum. In particular, it is possible that the sum is **not** well-defined. Thus we always require the sum to **converge absolutely**.

3 Expectation: Example Level 1

In the dice game with x dollars payout if x is rolled, the **expected earning** of a player is

$$\begin{aligned} E[X] &= \sum_{x \in S} x\,P(X = x) \\ &= 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + 3 \times \frac{1}{6} + \ldots + 6 \times \frac{1}{6} \\ &= \frac{7}{2}, \end{aligned}$$

so, to break-even, the casino owner needs to charge 3.5 dollars per game.

Suppose now that the payout is x^2 dollars if x is rolled.

The expected earning of a player is

$$
\begin{aligned}
E[X^2] &= \sum_{x \in S} x^2 \, P(X = x) \\
&= 1^2 \times \frac{1}{6} + 2^2 \times \frac{1}{6} + 3^2 \times \frac{1}{6} + \ldots + 6^2 \times \frac{1}{6} \\
&= \frac{91}{6} = 15\frac{1}{6}.
\end{aligned}
$$

so, to break-even, the casino owner needs to charge $15\frac{1}{6}$ dollars per game.

4 Expectations: Properties

Theorem 1. *The expectation E satisfies the following properties:*

- *If c is a constant, then $E[c] = c$.*
- *If c is a constant and u is a function, then*

$$E[cu(X)] = cE[u(X)].$$

- *If u_1, u_2 are functions, then*

$$E[u_1(X) + u_2(X)] = E[u_1(X)] + E[u_2(X)].$$

- *If X_1 and X_2 are independent, then*

$$E[X_1 X_2] = E[X_1]E[X_2].$$

5 Expectations: Example Level 2

Suppose now that the payout is $6x^2 + 13$ dollars if x is rolled. The expected earning of a player is

$$
\begin{aligned}
E[6X^2 + 13] =& 6\,E[X^2] + 13 \\
=& 6 \times \left(\frac{91}{6}\right) + 13 \\
=& 104.
\end{aligned}
$$

so, to break-even, the casino owner needs to charge 104 dollars per game.

6 Geometric random variable

Suppose that an experiment has probability of success $p \in (0, 1)$ and probability of failure $q = 1 - p$.

The experiment is repeated independently until the first success occurs.

Let X be the first time the success occurs.

This random variable is called the **geometric random variable**, and its distribution is called the geometric distribution.

The support of the geometric random variable X is

$$\{\text{the positive integers}\} \;=\; \{1, 2, 3, 4, \ldots\}$$

7 Geometric random variable: pmf

The pmf for the geometric random variable X is

$$f(1) = p; \quad f(2) = qp; \qquad f(3) = q^2p,$$

In general,

$$f(n) = q^{n-1}p \qquad \text{for } n \in \{1, 2, 3, 4, \dots\}.$$

8 Geometric random variable: Expectation

The expectation for the geometric random variable X is

$$
\begin{aligned}
E[X] =& \sum_{n=1}^{\infty} n f(n) \\
=&(1)p + (2)qp + (3)q^2p + (4)q^3p + \ldots \\
=&p\left(1 + 2q + 3q^2 + 4q^3 + \ldots\right) \\
=&p\frac{1}{(1-q)^2} \qquad \text{(why?)} \\
=&p\frac{1}{p^2} = \frac{1}{p}.
\end{aligned}
$$

Intuitively, if $p = 1/100$, then you would expect that $E[X] = 100$ trials are needed until you observe a success.

The answer to why why why is

$$
\begin{aligned}
\frac{1}{1-q} &= 1 + q + q^2 + q^3 + q^4 + \ldots \qquad \text{(from 31A)} \\
\frac{\partial}{dq}\frac{1}{1-q} &= \frac{\partial}{dq}\left(1 + q + q^2 + q^3 + q^4 + \ldots\right) \\
\frac{1}{(1-q)^2} &= 0 + 1 + 2q + 3q^2 + 4q^3 + \ldots
\end{aligned}
$$