Math 170E

Lecture Notes Section 1.3 [*,†]

Conditional probability

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NOTE: Materials that appear in the textbook but do not appear in the lecture notes might still be tested. Please send me an email if you find typos.

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[†] This notes is based on March Boedihardjo's and Jamie Haddock's notes from the past quarters, and I would like to thank them for their generosity. "*Nanos gigantum humeris insidentes* (I am but a dwarf standing on the shoulders of giants)".

1 Conditional probability: definition

The probability that an event A occurs **assuming that** another event B **has already happened** is given by

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)},$$

provided that $P(B) > 0$. This is called the **conditional probability** of A given B.

2 Conditional probability: Example Dice

Throw two dice. Given that the first die is 3, what is the probability that the sum of two dice equals 8?

Intuitive answer (not rigorous):

- Since the first die is 3, for the sum to be equal to 8, the second die must be 5.
- On the other hand, the second die can be any number from 1 to 6, each of them equally likely.
- Thus the probability for the event (sum equal to 8, given the first die is 3) is $\frac{1}{6}$.

3 Conditional probability: Answer Dice

Let B be the event that first throw is 3:

$$B := \{(3,1),(3,2),(3,3),(3,4),(3,5),(3,6)\}.$$

Let A be the event that sum of two dice is 8:

$$A := \{(2,6),(3,5),(4,4),(5,3),(6,2)\}.$$

Then $A \cap B = \{(3,5)\}$. Hence we have

$$P(A \cap B) = \frac{1}{36}; \qquad P(B) = \frac{1}{6};$$

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{1/36}{1/6} = \frac{1}{6}.$$

4 Multiplication rule

Theorem 1. *The probability $P(A \cap B)$ can be computed in two ways,*

$$P(A \cap B) \quad = \quad P(A)P(B \mid A),$$

provided $P(A) > 0$, or by

$$P(A \cap B) \quad = \quad P(B)P(A \mid B),$$

provided $P(B) > 0$.

One consequence of the multiplication rule is the Bayes theorem.

Theorem 2. *We have*

$$P(A \mid B) \quad = \quad \frac{P(A)}{P(B)}\, P(B \mid A),$$

provided $P(A), P(B) > 0$.

5 Conditional probability: Example Covid

Prevalence of Covid-19 in Duckburg is 6% in the general population.

The PCR test for Covid-19 has 0.5% false positive rate, and 20% false negative rate.

What is the probability that a person has the disease, given that they have tested positive?

6 Conditional probability: Answer Covid

Let D be the event that a person has Covid.

Let Y be the event that a person is tested positive.

The question is asking for $P(D \mid Y)$.

- The probability of having the disease if 6/100:

$$P(D) \;=\; \frac{6}{100}.$$

- The test has 0.5% false positive rate:

$$P(Y \mid D') \;=\; \frac{5}{1000}.$$

- The test has 20% false negative rate:

$$P(Y \mid D) \;=\; \frac{80}{100}.$$

Then we have

$$
\begin{aligned}
P(Y) &= P(Y \cap D) \ + \ P(Y \cap D') \\
&= P(D)\,P(Y \mid D) \ + \ P(D')\,P(Y \mid D') \\
&= \frac{6}{100}\frac{80}{100} \ + \ \frac{94}{100}\frac{5}{1000} \ = \ \frac{527}{10000}.
\end{aligned}
$$

So the answer to our question is (BT)

$$
\begin{aligned}
P(D \mid Y) &= \frac{P(D)}{P(Y)}\,P(Y \mid D) \\
&= \frac{\frac{6}{100}}{\frac{527}{10000}}\,\frac{80}{100} \\
&= \frac{480}{527} \quad \approx \quad 91\%.
\end{aligned}
$$

What is the probability that a person has the disease, given that they have tested negative? This question is asking for $P(D \mid Y')$.

We have

$$P(Y') \quad = \quad 1 - P(Y) \quad = \quad 1 - \frac{527}{10000} \quad = \quad \frac{9473}{10000},$$

and

$$P(Y' \mid D) \quad = \quad 1 - P(Y \mid D) \quad = \quad 1 - \frac{80}{100} \quad = \quad \frac{20}{100}.$$

So the answer to our question is (BT)

$$\begin{aligned} P(D \mid Y') &= \frac{P(D)}{P(Y')}\, P(Y' \mid D) \\ &= \frac{\frac{6}{100}}{\frac{9473}{10000}}\, \frac{20}{100} \\ &= \frac{120}{9473} \quad \approx \quad 1.26\%. \end{aligned}$$

7 Conditional probability: Example Apple

Apple wants to know if a previous customer will buy the Applewatch. They know that:

- 15% of the population owns only an iPhone;
- 2% of the population owns only an iPad;
- 3% of the population owns both.

The probability a previous customer will buy Applewatch,

- given that one only owns an iPhone now: 0.15;
- given that one only owns an iPad now: 0.10;
- given that one owns both products now: 0.05.

What is the probability that a previous customer buys an Applewatch?

8 Conditional probability: Answer Apple

Split the group of people into four types:

- A is the group who owns an iPhone only;
- B is the group who owns an iPad only;
- C is the group who owns both;
- D is the group of people buying the Applewatch.

The question is asking the probability of buying an Applewatch, given the person is a previous customer:

$$P(D \mid A \cup B \cup C).$$

The question gives us

$$P(A) = 0.15, \qquad P(B) = 0.02, \qquad P(C) = 0.03,$$

$$P(D \mid A) = 0.15, \qquad P(D|B) = 0.10, \qquad P(D \mid C) = 0.05.$$

Hence we have (BT)

$$\begin{aligned}
P(D \mid A \cup B \cup C) \quad &= \quad \frac{P(D \ \cap \ (A \cup B \cup C))}{P(A \cup B \cup C)} \\
= \quad & \frac{P(D \cap A) \ \cup \ (D \cap B) \ \cup \ (D \cap C))}{P(A \cup B \cup C)} \\
= \quad & \frac{P(D \cap A) + P(D \cap B) + P(D \cap C)}{P(A) + P(B) + P(C)} \\
= \quad & \frac{P(D \mid A)\, P(A) + P(D \mid B)\, P(B) + \ P(D \mid C) P(C)}{P(A) + P(B) + P(C)} \\
= \quad & \frac{0.15 \times 0.15 + 0.02 \times 0.10 + 0.03 \times 0.05}{0.15 + 0.02 + 0.03} \\
= \quad & \frac{0.026}{0.2} = 0.13.
\end{aligned}$$

Conditional probability is not an easy-to-understand concept, and even famous mathematicians have made mistakes regarding this concept!

Read the wikipedia's article on Monty Hall problem to find out more.