Math 170E

Lecture Notes Section 1.2 [*][†]

Methods of enumeration

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NOTE: Materials that appear in the textbook but do not appear in the lecture notes might still be tested. Please send me an email if you find typos.

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[†]This notes is based on March Boedihardjo's and Jamie Haddock's notes from the past quarters, and I would like to thank them for their generosity. "*Nanos gigantum humeris insidentes* (I am but a dwarf standing on the shoulders of giants)".

1 Multiplication principle: Example

Suppose that we want to elect a President and a Vice President from Alice, Bob, Charlie, David. How many ways can we do this?

2 Multiplication principle: Theorem

Theorem 1. *Suppose that the first experiment* E_1 *has* n_1 *outcomes, and the second experiment* E_2 *has* n_2 *outcomes, then the composite experiments* E_1E_2 *has* $n_1 \times n_2$ *outcomes.*

3 Multiplication principle: Answer

- There are 4 choices for the President;
- After the President is chosen, there are always 3 choices left for the Vice President, *regardless* of who the President is.

By multiplication principle, there are $4 \times 3 = 12$ outcomes for this experiments.

4 Permutations: Example

Suppose that Alice, Bob, Charlie, David are prisoners and we want to figure out an order for which they will be released from prison. How many ways can we do this?

5 Permutations: Theorem

Theorem 2. *There are $n!$ ways to arrange n objects into a sequence, where*

$$n! \;:=\; n \times (n-1) \times \ldots \times 2 \times 1.$$

Each arrangement is called a ***permutation*** *of the objects.*

6 Permutations: Proof

- There are n choices for the first element;
- After the first element is chosen, there are $n-1$ elements left to be chosen to be the second element, *regardless* of which element is chosen as the first.
- After the first and second element are chosen, there are $n-2$ elements left for the third element.
- $\cdots$
- After $n-1$ elements are chosen, there is 1 element left to be chosen to be the last element.

By multiplication principle, there are $n \times (n-1) \times \ldots \times 2 \times 1$ outcomes.

7 Permutations: Answer

The number of distinct prison release orders for Alice, Bob, Charlie, David is

$$4! = 4 \times 3 \times 2 \times 1 = 24.$$

8 Sampling without replacement: Example

How many ways can we select a president, a vice president, a secretary, and an errand-boy from a group of 11 people?

9 Sampling without replacement: Theorem

Sampling without replacement is when

- We select r elements from n different objects;
- We order them into a sequence;
- An object is **not replaced** after it has been selected.

Theorem 3. *There are* $\mathbf{{}_nP_r}$ *ways to perform sampling without replacement, where*

$${}_nP_r = \frac{n!}{(n-r)!} = n\times(n-1)\times\ldots\times(n-r+2)\times(n-r+1).$$

This is proved by multiplication principle, using the same idea as for permutations.

10 Sampling without replacement: Answer

The number of ways of selecting a president, a vice president, a secretary, and an errand-boy from a group of 11 people is

$$ {}_{11}P_4 = 11 \times 10 \times 9 \times 8 = \frac{11!}{7!} = 7920 . $$

11 Sampling with replacement: Theorem

Sampling with replacement is when

- We select r elements from n different objects;
- We order them into a sequence;
- An object is **replaced** after it has been selected.

Theorem 4. *There are* $\mathbf{n^r}$ *ways to perform sampling with replacement.*

This is again proved by multiplication principle.

12 Sampling with replacement: Example

California state license plates are of the form 1 X X X 1 1 1 (i.e., the 1st, 5th-7th character are nonzero digits and the 2nd-4th character are letters). How many distinct license plates can the state issue?

Answer. We first sample four digits with replacements, for which there are 9^4 outcomes.
We then sample three letters with replacements, for which there are $(26)^3$ outcomes.
By multiplication principle, the number of distinct license plates is

$$9^4 \times (26)^3 \;=\; 115316136\,,$$

as desired. $\square$

13 Counting without order: Example

How many five-card hands can we form from a deck of 52 playing cards?

14 Counting without order: Theorem

Counting when order does not matter is when

- We select r elements from n different objects;
- **The order does not matter** (i.e., take the set formed by these r elements);
- An object is **not replaced** after it has been selected.

Theorem 5. *If you are being asked to choose r elements from n different objects and the order does* NOT *matter, then there are* $\mathbf{{}_nC_r}$ *ways to do this, where*

$$ {}_nC_r := \binom{n}{r} = \frac{{}_nP_r}{r!} = \frac{n!}{r!(n-r)!}. $$

*These numbers are the famous **binomial coefficients**.*

15 Counting without order: Proof

We first perform counting **without order**, where the number of outcomes is ${}_nC_r$. (At this stage we still don't know the value of ${}_nC_r$ yet.)

Then, given an (**unordered**) set $\{a_1, \ldots, a_r\}$, the number of ways to arrange it into an **ordered** sequence is number of permutations $r!$.

By multiplication principle, the number of **all ordered sequences** is ${}_nC_r \times r!$.

On the other hand, this is **sampling without replacement**. This is sampling without replacement, so number of outcomes is $\frac{n!}{(n-r)!}$.

Combining these two observations,

$$ {}_nC_r \times r! \quad = \quad \frac{n!}{(n-r)!}. $$

16 Counting without order: Example

The number of possible five-card hands that can be formed from a deck of 52 playing cards is

$$\begin{aligned} {}_{52}C_5 &= \binom{52}{5} = \frac{52!}{5!\;47!} = \frac{52 \times 51 \times 50 \times 49 \times 48}{1 \times 2 \times 3 \times 4 \times 5} \\ &= \frac{311875200}{120} \\ &= 2598960\,. \end{aligned}$$

17 Binomial theorem

Theorem 6.

$$
\begin{aligned}
(a+b)^n &= \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r \\
&= a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \ldots + \binom{n}{n} b^n.
\end{aligned}
$$

For example, (BT)

$$
\begin{aligned}
(a+b)^2 &= a^2 + \binom{2}{1} ab + \binom{2}{2} b^2 = a^2 + 2ab + b^2 \\
(a+b)^3 &= a^3 + \binom{3}{1} a^2 b + \binom{3}{2} ab^2 + \binom{3}{3} b^3 \\
&= a^3 + 3a^2 b + 3ab^2 + b^3.
\end{aligned}
$$

18 Multinomial theorem

For all $n \geq 0$ and all nonnegative integers $n_1, \ldots, n_s$ such that $n_1 + \ldots + n_s = n$, the **multinomial coefficient** is

$$\binom{n}{n_1, n_2, \ldots, n_s} := \frac{n!}{n_1!\; n_2!\; \ldots\; n_s!}.$$

Theorem 7. *For any real numbers* $a_1, a_1, \ldots, a_s$,

$$(a_1 + \ldots + a_s)^n = \sum_{n_1, n_2, \ldots, n_s} \binom{n}{n_1, n_2, \ldots, n_s} a_1^{n_1} a_2^{n_2} \ldots a_s^{n_s},$$

where $n_1, n_2, \ldots, n_s$ *ranges over all nonnegative integers such that* $n_1 + n_2 + \ldots + n_s = n$.

19 Counting objects that are of two types: Example

A basketball player takes 50 free throw shots and misses exactly two. How many ways could he have done this?

20 Counting objects that are of two types: Theorem

Theorem 8. *Suppose that you are handed n objects, r of type A and $n - r$ of type B. Then the number of ordered sequences you can make out of these objects is ${}_nC_r$.*

21 Counting objects that are of two types: Proof

We first select r elements from the set $\{1, \ldots, n\}$, where order does not matter, without replacement (counting without order).

For example, when $r = 3$ and $n = 7$, we select the set

$$\{1, 3, 6\}.$$

We then create the ordered sequence of two types as follows:

- The i-th element is of type **A** if i is **contained** in the set of selected r elements;
- The i-th element is of type **B** if i is **not contained** in the set of selected r elements;

For example, from the set $\{1, 3, 6\}$ above, we create the sequence

$$ABABBAB.$$

Hence the number of **ordered sequences of two types** is the number of **counting without order**, which is equal to ${}_nC_r$.

22 Counting objects that are of two types: Answer

The number of ways for a basketball player to take 50 free throw shots and miss exactly two is

$$ {}_{50}C_2 = \binom{50}{2} = \frac{50 \times 49}{2 \times 1} = 1225. $$