Math 170E

Lecture Notes Section 1.1 [*][†]

Introduction to probability

Instructor: Swee Hong Chan

NOTE: Materials that appear in the textbook but do not appear in the lecture notes might still be tested. Please send me an email if you find typos.

[*]Version date: Wednesday 6$^{\text{th}}$ January, 2021, 15:00.

[†]This notes is based on March Boedihardjo's and Jamie Haddock's notes from the past quarters, and I would like to thank them for their generosity. "*Nanos gigantum humeris insidentes* (I am but a dwarf standing on the shoulders of giants)".

1 Probability

Why should we study probability?

"*There is nothing certain; but the uncertain.*"

That is why, in order to not let *uncertainty* being in control of our life, we need to understand the true nature of *uncertainty* and use this knowledge to achieve our life's objectives.

2 Probability: example

When you throw a six-sided dice, there are six possible outcomes:

- With probability $\frac{1}{6}$, we have number 1 as the outcome;
- With probability $\frac{1}{6}$, we have number 2 as the outcome;
- $\ldots$;
- With probability $\frac{1}{6}$, we have number 6 as the outcome.

These are all the six outcomes, and the sum of the probabilities is equal to 1.

3 Outcome space and events

The **outcome space** is the collection of all possible outcomes of a random experiment, usually denoted by S.

In the dice example, the random experiment is a dice-throw, and the outcome space is $\{1, 2, 3, 4, 5, 6\}$.

An **event** is a subset of outcomes. An event A **has occurred** if the outcome of the random experiment **is contained** in A.

Here are some possible events for the dice example:

- $A := \{1, 2, 4, 5\}$;
- $B := \{\text{the outcome is even}\} = \{2, 4, 6\}$.
- $C := S = \{1, 2, 3, 4, 5, 6\}$.
- $D := \{\} = \varnothing \quad$ (empty set).

4 Relative frequency

Suppose that you repeat a random experiment n times.

The **frequency** of an event A is the number of times the event actually occured, and is denoted by $N(A)$.

The **relative frequency** of A in n experiments is $N(A)/n$.

For example, suppose that you threw a dice $n = 10$ times:

$$1\ ,\ 1\ ,\ 3\ ,\ 5\ ,\ 6\ ,\ 3\ ,2\ ,\ 3\ ,\ 4\ ,\ 4\ .$$

Then the relative frequencies are:

$$\begin{aligned}
\frac{N(\{1\})}{n} &= \frac{2}{10}; \\
\frac{N(\{\text{even}\})}{n} &= \frac{4}{10}; \\
\frac{N(S)}{n} &= \frac{10}{10} = 1; \\
\frac{N(\{\})}{n} &= \frac{0}{10} = 0.
\end{aligned}$$

5 Probability

The **probability** of an event A is the limit of $N(A)/n$ as n grows larger and larger.

The probability of A will be denoted by $P(A)$.

For example, for the dice example:

$$\begin{aligned} P(\{1\}) &= \frac{1}{6}; \\ P(\{\text{even}\}) &= \frac{3}{6}; \\ P(S) &= 1; \\ P(\{\}) &= 0. \end{aligned}$$

6 Rules that P needs to satisfy

The **probability function** P is a function that assigns to each event A a real number $P(A)$.

Any probability function satisfies:

- $P(A) \geq 0$ for any event A;
- $P(S) = 1$;
- Let $A_1, A_2, \ldots A_k$ be k events such that $A_i \cap A_j = \varnothing$ whenever $i \neq j$. Then

  $$P(A_1 \cup A_2 \cup \ldots \cup A_k) = P(A_1) + P(A_2) + \ldots + P(A_k).$$

7 Uniform probability measure

Suppose $S = \{e_1, e_2, \ldots, e_m\}$. All outcomes of S are **equally likely** if

$$P(\{e_1\}) = P(\{e_2\}) = \ldots = P(\{e_m\}).$$

In this case, we say P is the **uniform probability measure** on S.

8 Mutually exclusive and exhaustive events

Events $A_1, A_2, \ldots, A_k$ are **mutually exclusive** if $A_i \cap A_j = \varnothing$ whenever $i \neq j$.

Events $A_1, A_2, \ldots, A_k$ are **exhaustive** if

$$A_1 \cup A_2 \cup \ldots \cup A_k \;=\; S.$$

Exercise: Draw the Venn diagram to illustrate these two concepts.

9 Complement

The **complement** of an event A is the set containing all elements not in A.

The complement of A will be denoted A'.

Theorem 1. $P(A) = 1 - P(A')$.

Proof. We have (BT)

$$\begin{aligned} P(A) + P(A') &= P(A \cup A') \quad \text{(rule of mutual exclusion)} \\ &= P(S) \quad \text{(definition of complement)} \\ &= 1\,. \end{aligned}$$

This is equivalent to the theorem. $\square$

10 Probability for empty set

Theorem 2. $P(\varnothing) = 0$.

Proof. We have

$$\begin{aligned} P(\varnothing) &= P(\text{complement of } S) \\ &= 1 - P(S) \\ &= 1 - 1 \quad = \quad 0, \end{aligned}$$

as desired. $\square$

11 Probability of larger events

Theorem 3. *If $A \subseteq B$, then $P(A) \leq P(B)$.*

Proof. We write

$$B - A \ := \ \{\text{elements of } B \text{ not in } A\,\}.$$

We have (BT)

$$\begin{aligned} P(A) \quad &\leq \quad P(A) + P(B - A) \\ &= \quad P(A \cup (B - A)) \quad \text{(rule of mutual exclusion)} \\ &= \quad P(B) \quad (\text{since } A \subseteq B), \end{aligned}$$

as desired. □

12 Inclusion-exclusion principle (two elements)

Theorem 4.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Proof. By the rule of mutual exclusion: (BT)

$$\begin{aligned} P(A) &= P(A - B) + P(A \cap B); \\ P(B) &= P(B - A) + P(A \cap B). \end{aligned}$$

Adding the two equalities above:

$$\begin{aligned} & P(A) + P(B) \\ = \quad & P(A - B) + P(A \cap B) \; + \; P(B - A) + P(A \cap B) \\ = \quad & P(A \cup B) + P(A \cap B), \end{aligned}$$

which is equivalent to the theorem. □

13 Inclusion-exclusion principle (many elements)

Theorem 5.

$$\begin{aligned} P(A \cup B \cup C) = &P(A) + P(B) + P(C) \\ &- P(A \cap B) - P(A \cap C) - P(B \cap C) \\ &+ P(A \cap B \cap C). \end{aligned}$$

Theorem 6. *For general events* $A_1, \ldots, A_k$,

$$P(A_1 \cup \ldots \cup A_k) \quad = \quad \sum_{m=1}^{k} (-1)^{m+1} \sum_{i_1, \ldots, i_m} P(A_{i_1} \cap \ldots \cap A_{i_m}),$$

where $i_1, \ldots, i_m$ *ranges over all integers satisfying* $1 \leq i_1 \leq \ldots i_m \leq k$.