Math 184

Lecture Notes Section 5.5 *

Instructor: Swee Hong Chan

NOTE: The notes is a summary for materials discussed in the class and is not supposed to substitute the textbook. In particular, the proofs here might omit some details for brevity, and are not supposed to be how you write proofs in the exam. Please refer back to the textbook when studying for exams; materials that appear in the textbook but do not appear in the lecture notes might still be tested. Please send me an email if you find typos.

1 Rooted trees

Example 1. You are going to the yearly family gathering, but you have trouble remembering the honorifics for your relatives. You therefore use the following table as a cheatsheet:

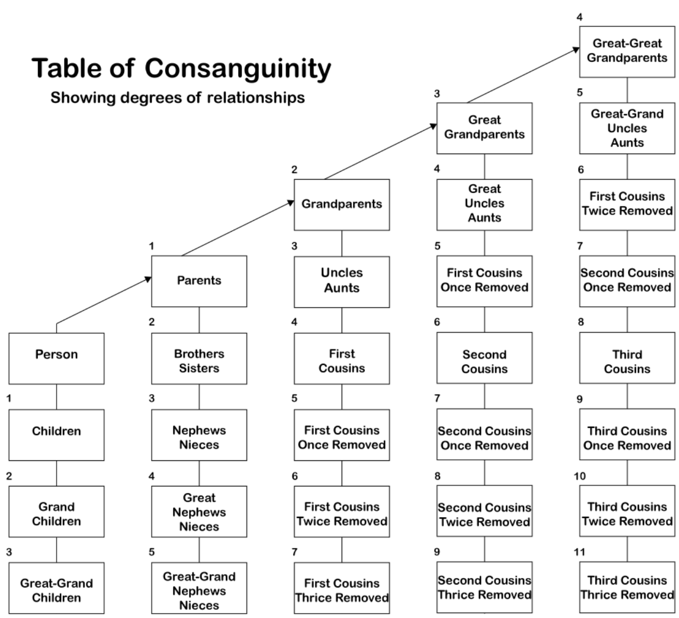

Figure 1: Table of consanguinity showing degrees of relationships. The number next to each box in the table indicates the degree of relationship relative to the given person.

△

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The table above is an example of a rooted plane tree, with your great-great grandparent as the root. The following is the definition of rooted plane tree, given as a recursive construction.

Definition 2 (Rooted plane tree)**.** A *rooted plane tree* is a tree drawn in the 2-dimensional plane with a distinguished vertex R as the *root*, so that the neighbors $n_1, \dots, n_k$ of R are all roots of smaller plane trees. △

Example 3. There are exactly five rooted plane trees with 4 vertices, as drawn in Figure 2. △

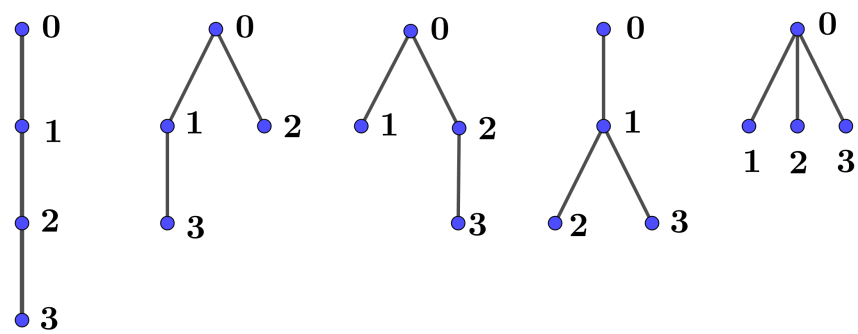

Figure 2: Five rooted plane trees with 4 vertices, with labels taken from $\{0, 1, 2, 3\}$

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Remark 4. We adopt the convention that a rooted plane tree of n vertices has vertices labeled by $\{0, 1, 2, \dots, n-1\}$, and they are labeled based on the breadth-first search algorithm. That is to say,

- The root is labeled by 0;
- The k children (neighbor) of the root is labeled $1, 2, \dots, k$, from left to right,
- the ℓ grandchildren of the root is labeled $k+1, k+2, \dots k+\ell$, again from left to right,
- etc. △

We would now count the number of rooted plane trees b_n with n vertices by using the generating function argument.

Theorem 5. *Let n be a positive integer. The number of rooted plane trees on n vertices is*

$$b_n = \frac{1}{n}\binom{2n-2}{n-1},$$

which is the $n-1$-th Catalan number.

Proof. Let $B(x) = \sum_{n=0}^{\infty} b_n x^n$ be the corresponding generating function, where we adopt the convention that $b_0 = 0$. Now note that removing the root (the vertex 0) of a rooted plane tree, we get a sequence of (smaller) rooted plane trees. This is the same as splitting $[n-1]$ (the remaining vertices) into disjoint blocks,

and at each block we apply the construction of plane rooted trees. This is the scenario of the composition formula for ordinary generating functions, so we conclude that

$$B(x) = x \cdot \frac{1}{1 - B(x)},$$

where the term x is because the construction is applied to $[n]$ rather than $[n-1]$, and the term $\frac{1}{1-B(x)}$ comes from the composition formula for generating function. By doing some algebraic manipulations to the equation,

$$\begin{aligned} B(x)(1 - B(x)) &= x \\ B(x) \ - \ B(x)^2 &= x \\ 0 &= B(x)^2 - B(x) + x. \end{aligned}$$

This is a quadratic equation for $B(x)$, and therefore we can solve it as such. The solution is

$$B(x) \ = \ \frac{1 - \sqrt{1-4x}}{2}.$$

Now remember that, by the general binomial theorem,

$$\begin{aligned} \sqrt{1-4x} \ &= \ (1 + (-4x))^{1/2} \ = \ 1 + \sum_{n=1}^{\infty} \binom{1/2}{n} (-4x)^n \\ &= 1 + \sum_{n=1}^{\infty} \frac{\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{3}{2}\right)\cdots\left(\frac{-(2n-3)}{2}\right)}{n!}(-4x)^n \\ &= 1 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\frac{1\cdot 3 \ldots (2n-3)}{2^n}}{n!}(-1)^n 4^n x^n \\ &= 1 - \sum_{n=1}^{\infty} \frac{1 \cdot 3 \ldots (2n-3)}{n!} 2^n x^n \\ &= 1 - \sum_{n=1}^{\infty} \frac{1 \cdot 2 \cdot 3 \cdot 4 \ldots (2n-3) \cdot (2n-2)}{n!(2 \cdot 4 \cdot 6 \ldots (2n-2))} 2^n x^n \\ &= 1 - \sum_{n=1}^{\infty} \frac{(2n-2)!}{n!(n-1)!2^{n-1}} 2^n x^n \\ &= 1 - 2\sum_{n=1}^{\infty} \frac{(2n-2)!}{n!(n-1)!} x^n \\ &= 1 - 2\sum_{n=1}^{\infty} \frac{1}{n} \frac{(2n-2)!}{(n-1)!(n-1)!} x^n \\ &= 1 - 2\sum_{n=1}^{\infty} \frac{1}{n} \binom{2n-2}{n-1} x^n. \end{aligned}$$

Plugging this into the formula for $B(x)$, we conclude that

$$B(x) \ = \ \frac{1 - \sqrt{1-4x}}{2} \ = \ \frac{1}{2}\left(1 - \left[1 - 2\sum_{n=1}^{\infty} \frac{1}{n}\binom{2n-2}{n-1} x^n\right]\right) \ = \ \sum_{n=1}^{\infty} \frac{1}{n}\binom{2n-2}{n-1} x^n.$$

We therefore conclude that

$$b_n = \frac{1}{n}\binom{2n-2}{n-1},$$

as desired. $\square$

Remark 6. It is not a coincidence that the number of plane rooted trees with n vertices is equal to the $n-1$-th Catalan number, which is also the number of the northeastern lattice paths from $(0,0)$ to $(n-1, n-1)$. These two sets are in a (natural) bijection to each other, and the bijection can be found in Theorem 5.28 in the textbook. $\triangle$

2 Eulerian graph

Example 7 (Königsberg bridge problem)**.** The city of Königsberg in Prussia (now Kaliningrad, Russia) was set on both sides of the Pregel River, and included two large islandsKneiphof and Lomsewhich were connected to each other, or to the two mainland portions of the city, by seven bridges. The problem was to devise a walk through the city that would cross each of those bridges once and only once.

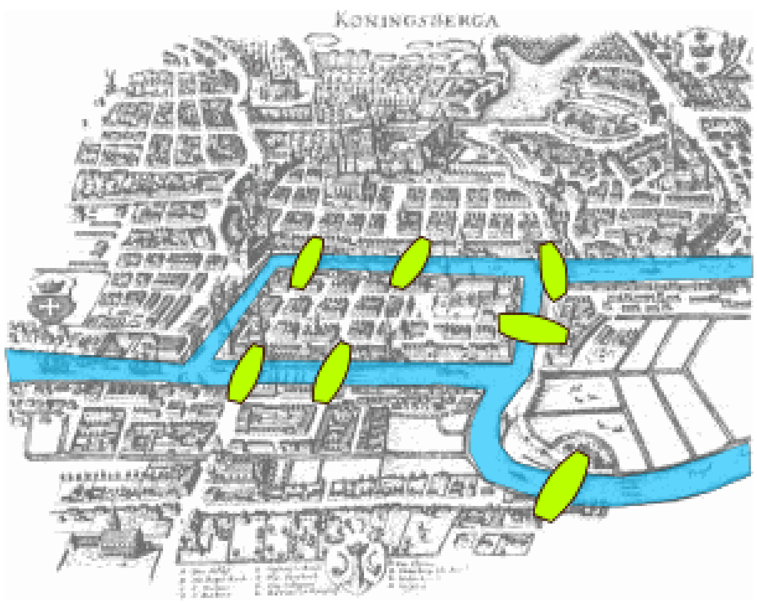

Figure 3: Map of Königsberg in Euler's time showing the actual layout of the seven bridges, highlighting the river Pregel and the bridges.

This problem was solved by Euler in 1736, by interpreting the four landed areas as vertices and the seven bridges as edges of the graph in Figure 4. The problem is now to draw the given graph in one go without ever lifting your pencil up until the end. The graph that satisfies this property is now called *Eulerian graphs*, defined rigorously below. $\triangle$

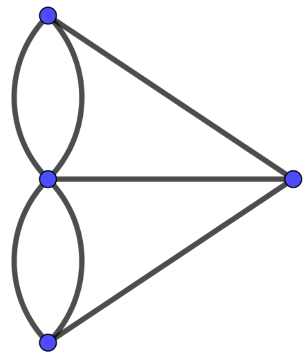

Figure 4: The graph obtained from the map of Königsberg by translated landed areas into vertices and the bridges into edges.

Definition 8. A graph G is *Eulerian* if there exists a walk $e_0 e_1 e_2 \dots e_n$ in the graph that starts and ends at the same vertex, and uses every edge of G exactly once. △

Theorem 9. *A graph G is Eulerian if and only if G is connected and all vertices of G have even degree.*

Proof. Please read the textbook for a proof of this theorem, which uses induction. □

One consequence of the theorem is that the graph in Figure 4 is NOT Eulerian since all of its vertices have odd degrees. This gives an easy proof that the answer to the Königsberg bridge problem is NO.
As it turns out, the number of (simple) Eulerian graphs have no nice formulas. However, if we drop the condition that it is connected, then we can get a nice formula.

Definition 10. A simple graph G is *even* if all vertices of G have even degree. Equivalently, a simple graph G is even if all of its connected components are Eulerian graphs. △

Theorem 11. *For all positive integers n, the number of even graphs on vertex set $[n]$ is*

$$c_n = 2^{\binom{n-1}{2}}.$$

Proof. We prove the theorem by presenting a bijecton f from the set A of all simple graphs on $[n-1]$ onto the set B of all even graphs on $[n]$. The bijection f can be described as follows. Given any simple graph G on $[n-1]$, we do the following:

- Add a new vertex labeled by n into G;
- Connect n into each old vertex of G that has odd degree.

See Figure 5 for an illustration of this process.

This operation will turn the degrees of all old vertices of G into even numbers. It will also turn the degre of n into even, as G must have even number of vertices of odd degree by the handshaking lemma. Hence $f(G)$ is a simple even graph of $[n]$.

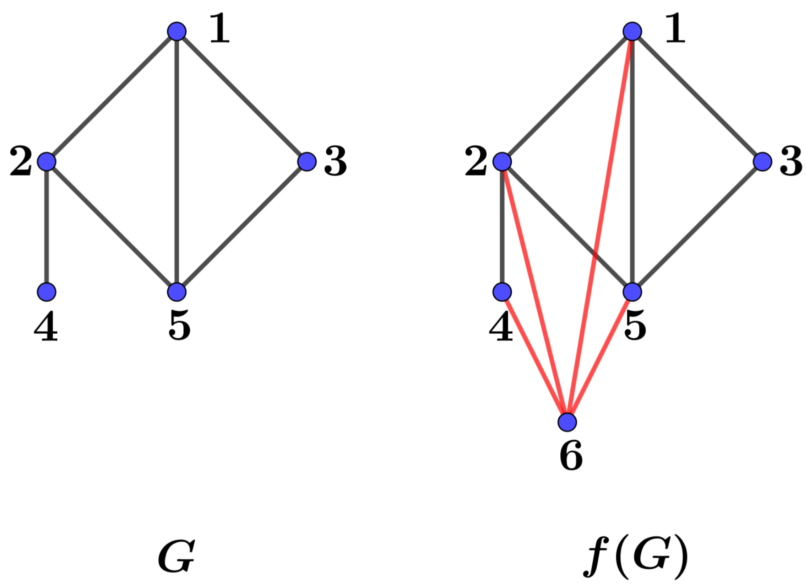

Figure 5: A graph G on $[5]$ turned into an even graph $f(G)$ on $[6]$.

We now show that f is a bijection by showing that each even graph H on $[n]$ has a unique preimage under f. To see this, we have that (by construction) $f^{-1}(H)$ can be obtained from H by omitting the vertex n and all the edges adjacent to n. This shows that f is a bijection, as desired. $\square$

Remark 12. Read other parts of Section 5.5 not covered in the notes! $\triangle$