Math 184

Lecture Notes Section 5.4 *

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NOTE: The notes is a summary for materials discussed in the class and is not supposed to substitute the textbook. In particular, the proofs here might omit some details for brevity, and are not supposed to be how you write proofs in the exam. Please refer back to the textbook when studying for exams; materials that appear in the textbook but do not appear in the lecture notes might still be tested. Please send me an email if you find typos.

1 Chromatic polynomial

Example 1. You are now hired by Friendly Instructor Map to color any given map so that no two adjacent countries have the same color. Here is one such coloring of the world map using four colors:

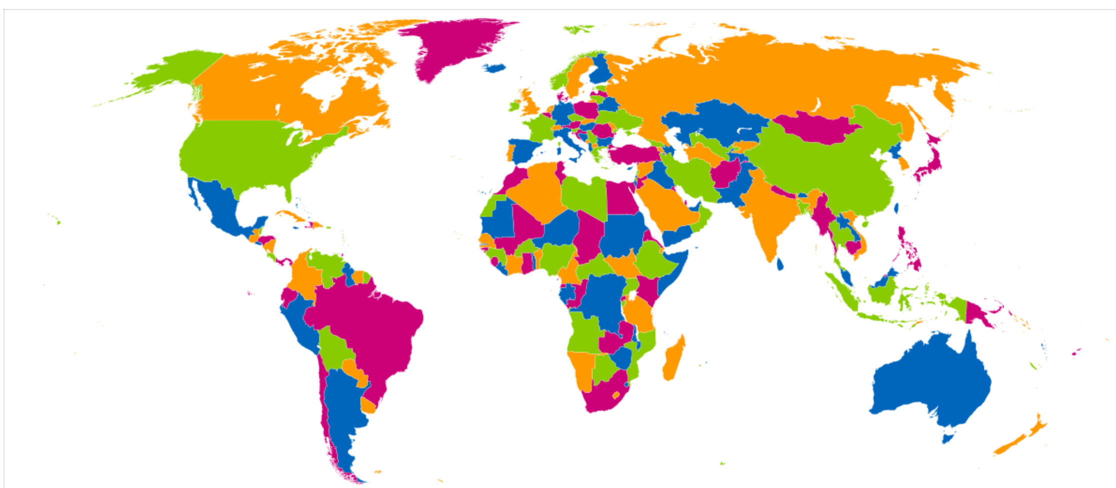

Figure 1: A coloring of the world map with four colors.

Note that you need at least four colors to color the world map above, as China, Kazakshtan, Mongolia, and Russia share the same border at a point[1]. △

The example above motivates the following very natural question. Is four colors always enough to color all possible maps? This question, which is simple to state, turns out to be a very hard problem! It took mathematicians 100 years and helps from computers to finally answer this question.

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[1]In reality, Mongolia and Kazakhstan do not share a border if one zoom into the border point using Google Maps.

Theorem 2 (Four color theorem)**.** *Given any map, no more than four colors are required to color the regions of the map so that no two adjacent regions have the same color.* □

Before the proof of the theorem was found, there were several different approaches proposed to solve the problem, and one of them is through studying the proper colorings of graphs.

Definition 3 (Proper (vertex) coloring)**.** A *proper coloring* of G is an assignment of colors to the vertices G so that no two adjacent vertices have the same color. That is to say, if x and y are two vertices of G such that $\{x, y\} \in E$, then x and y are colored differently. △

The following is a proper coloring of a ten-vertex graph (the Petersen graph) with three colors.

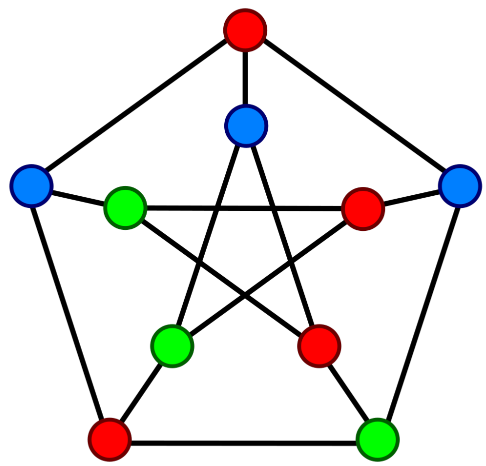

Figure 2: A proper coloring of the Petersen graph with three colors.

One thing we are interested in is the number of proper colorings of a given graph. This number is important enough that it gets its own name.

Definition 4 (Chromatic polynomial)**.** Let G be a graph. The *chromatic polynomial* $\chi_G : \mathbb{Z}_{>0} \to \mathbb{Z}$ is the function that maps the positive integer n to the number $\chi_G(n)$ of proper colorings of G using only the colors 1,2,. . . , n. Note that we do not have to use all these colors. △

Example 5. Let G be the simple complete graph on vertex set $[m]$. Then, for any positive integer n, we have

$$\chi_G(n) = n(n-1)(n-2)\ldots(n-m+1).$$

This is because we have n choices for the color of the first vertex, $n-1$ choices for the color of the second vertex, and so on. △

Example 6. Let G be the graph on vertex set $[m]$ with no edges. Then, for any positive integer n, we have

$$\chi_G(n) = n^m,$$

as there are n choices of color for every vertex. △

Remark 7 (Historical)**.** Chromatic polynomial was first studied by Birkhoff in 1912, in an attempt to prove the four color theorem. This is because proving the four color theorem can be reduced to solving a problem on graph coloring as follows: Given any map M, let G_M be the graph given by

$$\begin{aligned} V &= \text{set of countries in the map } M; \\ E &= \{\{x, y\} \in V \times V \mid x \text{ and } y \text{ are two countries sharing a border in the map } M\}. \end{aligned}$$

Then the four color theorem is equivalent to showing that $\chi_{G_M}(4) > 0$ for all map M.

Historically, this approach failed to prove the four color theorem. However, later mathematicians found many other applications for the chromatic polynomial that the chromatic polynomial arguably ended up being more important than the four color theorem that motivated its definition. △

Note that the function χ_G is called the chromatic *polynomial* instead of the chromatic *function*. This is because of the following theorem.

Theorem 8. *For any graph G, the function $\chi_G(n)$ is a polynomial function of n. Furthermore, the degree of this polynomial is equal to the number of vertices of G.*

Proof. Let m be the number of vertices of G. Now matter how large n is, no coloring of G uses more than m colors (as there are only m many vertices to color). So, in coloring G with n colors, we can first choose the number $i \in \{1, \ldots, m\}$ of colors that *are actually used* out of available n colors (this has $\binom{n}{i}$ choices), then color G using these i colors. It then follows from this argument that

$$\chi_G(n) = \sum_{i=1}^{m} \binom{n}{i} r_i$$

where r_i is the number of proper colorings of G with colors from $[i]$ that *actually use all i colors.* Now note that $\chi_G(n)$ is a polynomial of degree m, because the equation above can also be written as

$$\chi_G(n) = \sum_{i=1}^{m} [n(n-1)\ldots(n-m+1)] \frac{r_i}{i!},$$

which proves the theorem. □

2 Deletion-contraction recurrence

Perhaps the most important property of the chromatic polynomial is that it satisfies a recurrence called the *deletion-contraction* recurrence, defined as follows.

Definition 9 (Deletion)**.** Let G be a graph, and let e be an edge of G. The *deletion* of G, denoted by $G - e$, is the graph obtained from G by deleting the edge e. See Figure 3 for an example. △

Definition 10 (Contraction)**.** Let G be a graph, and let $e =$ be an edge of G. The *contraction* of G, denoted by G/e, is the graph obtained from G by merging the endpoints of e and then removing the resulting loops and/or multiple edges. See Figure 4 for an example. △

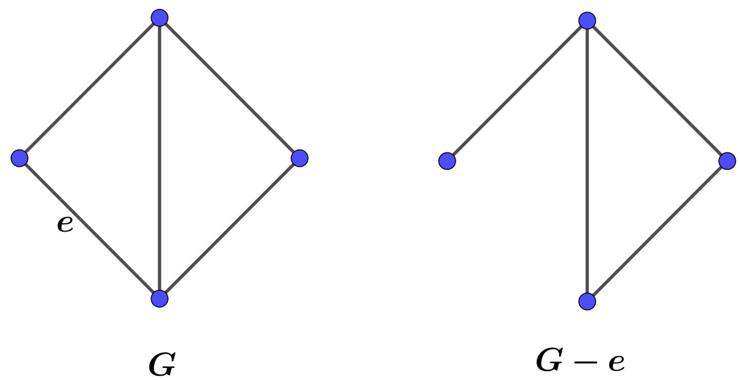

Figure 3: The deletion operation on a graph G with edge e.

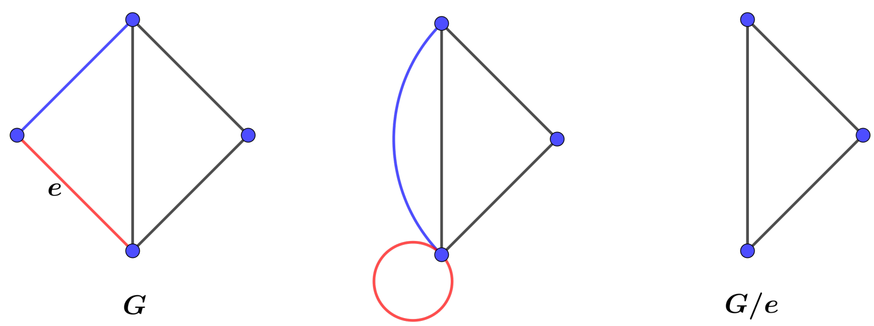

Figure 4: The contraction operation on a graph G with edge e.

Remark 11. The standard definition of contraction *removes the loop corresponding to e, but does NOT remove other resulting loops or multiple edges.* In particular, the result of a contraction operation might not always be a simple graph. This subtlety is important when one works with a more general polynomial called Tutte polynomial. However, in the scope of our course (focusing only on chromatic polynomials) we can remove these multiple edges and loops for simplicity. △

Theorem 12 (Deletion-contraction recurrence)**.** *For all nonnegative integers n, the following holds:*

1. *For any edge e of G,*

$$\chi_G(n) = \chi_{G-e}(n) - \chi_{G/e}(n);$$

2. *For any graph G and H,*

$$\chi_{G+H}(n) = \chi_G(n) \cdot \chi_H(n),$$

where $G + H$ is the disjoint union of G and H;

3. *For the graph K_1 with one vertex and no edges, we have*

$$\chi_{K_1}(n) = n.$$

Proof. 1. There are two types of proper colorings of $G - e$:

- Colorings where the colors of the two endpoints of e have different colors. These colorings are exactly the proper colorings of G, so the number of these colorings is $\chi_G(n)$.
- Colorings where the colors of the two endpoints of e have the same color. Then we can merge the two endpoints of G, and these colorings are exactly the proper colorings of G/e, so the number of these colorings is $\chi_{G/e}(n)$.

Based on the observation above, we conclude that

$$\chi_{G/e}(n) + \chi_G(n) \;=\; \chi_{G-e}(n),$$

which is equivalent to

$$\chi_G(n) \;=\; \chi_{G-e}(n) - \chi_{G/e}(n),$$

as desired.

This claim follows immediately from the product formula for counting from Section 1.

This claim follows immediately from the fact that there are exactly n ways to color a single vertex with n colors. □

As a matter of fact, if P is any polynomial that satisfies the three properties in Theorem 12, then $P = \chi_G$. This is actually THE reason why chromatic polynomial is so widely studied, as these recurrence turns out to be ubiquitous in mathematics. We include several examples of such appearances in the remarks below.

Remark 13. Three areas where chromatic polynomials (more accurately, deletion-contraction recurrence) surprisingly appear include:

- Knot theory, which is the field studying how to unravel knots[2]. The deletion-contraction recurrence appears in the definition of HOMFLY polynomial, which determines if the corresponding knot can be unraveled.
- Statistical physics, where the deletion-contraction recurrence appears in the definition of critical polynomials, which explains how complex patterns (e.g., fractal patterns in snow particles) can arise naturally from simple local interactions[3].

[2] Perhaps the most interesting way to illustrate the importance of this field is the story of Gordion Knot, for which any man who could unravel this knot was destined to become ruler of all of Asia. This knot was unraveled by Alexander The Great, who ended conquering parts of Asia.

[3] This is one of the research area that the friendly instructor is dwelling in

- Hyperplane arrangement theory, which is the field studying the geometry of intersections of (possibly different-dimensional) spaces. △

We now apply the deletion-contraction recurrence to compute the chromatic polynomial of a tree.

Lemma 14. *Let T_m be a tree with m vertices. Then the chromatic polynomial of T is equal to*

$$\chi_{T_m}(n) = n(n-1)^{m-1}.$$

Proof. We will prove the claim by induction on m. The base case $m = 1$ follows from $T_1 = K_1$, which gives us

$$\chi_{T_1}(n) = n.$$

Suppose now that the claim is true for $m-1$, we want to prove that the claim is true for m. Let v be a vertex of T with degree 1, and let e be the edge adjacent to v. It follows from the deletion contraction recurrence that

$$\chi_{T_m}(n) = \chi_{T_m - e}(n) - \chi_{T_m/e}(n).$$

See Figure 5 for an illustration of $T_m - e$ and T_m/e.

We now analyze the chromatic polynomial of T_m/e and $T_m - e$ separately:

- Note that T_m/e is a tree with $m-1$ vertices, so by the induction assumption,

$$\chi_{T_m/e}(n) = n(n-1)^{m-2}.$$

- Note that T_m/e is the disjoint union of T_m/e with the graph K_1 (which has one vertex and no edges). It then follows that

$$\chi_{T_m - e}(n) = \chi_{T_m/e}(n) \cdot \chi_{K_1}(n) = n(n-1)^{m-2}n.$$

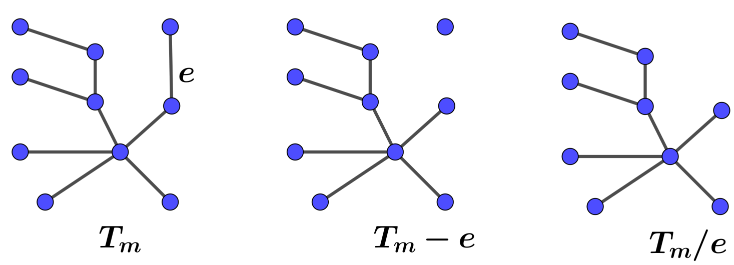

Figure 5: A tree T_m with m vertices, and the result of deletion and contraction of T_m by an edge e, respectively.

Combining these observations, we conclude that

$$\begin{aligned}\chi_{T_m}(n) \;=\; \chi_{T_m - e}(n) - \chi_{T_m/e}(n) \;&= n(n-1)^{m-2}n \;-\; n(n-1)^{m-2} \\ &= n(n-1)^{m-2}(n-1) \;=\; n(n-1)^{m-1},\end{aligned}$$

which proves the induction step. □

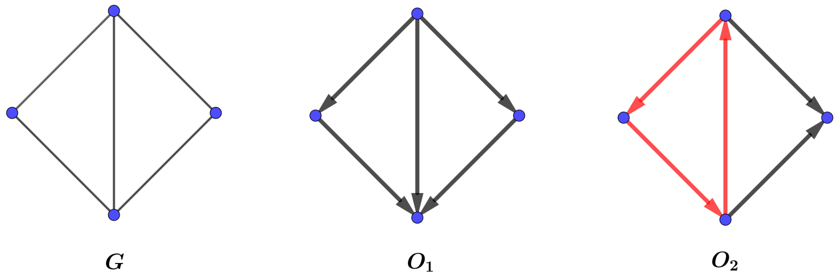

Figure 6: A graph G with two of its orientation O_1 and O_2. Note that O_1 is an acyclic orientation but O_2 is not an acyclie orientation (because of the red directed cycles).

3 Acyclic orientations

We now give a combinatorial meaning to the chromatic polynomial $\chi_G(n)$ when $n = -1$.

Definition 15 (Acyclic orientations)**.** An *orientation* of G is an assignment of a direction to each edge of G. An orientation is *acyclic* if no directed cycles are formed. See Figure 6 for examples. △

Theorem 16. *Let G be a graph with m vertices. Then the number of acyclic orientations of G is equal to*

$$(-1)^m \chi_G(-1).$$

Proof. Please read the textbook for the proof, which uses the observation that the number of acyclic orientations satisfies the deletion-contraction recurrence. □

Corollary 17. *Let T_m be a tree with m vertices. Then the number of acyclic orientations of T_m is equal to*

$$2^{m-1}.$$

Proof. We have

$$\begin{aligned} \text{Number of acyclic orientations of } T_m &= (-1)^m \chi_{T_m}(-1) \qquad \text{(by Theorem 16)} \\ &= (-1)^m(-1)((-1)-1)^{m-1} \qquad \text{(by Lemma 14)} \\ &= (-1)^{m-1}(-2)^{m-1} \\ &= 2^{m-1}, \end{aligned}$$

as desired.

Note that another way to prove the claim is by noting that since T_m has $m-1$ edges, it then has 2^{m-1} orientations (2 choices for every edge). Since T_m has no cycles, all these 2^{m-1} orientations are acyclic, and the claim now follows. □

In fact, the chromatic polynomial encodes more information than just the number of acyclic orientations of G, which we state explicitly in the next theorem.

Theorem 18. *Let G be a graph with m vertices, and let n be a positive integer. Then the number of pairs (f, A) such that*

- *A is an acyclic orientation of G; and*
- *$f : V \to [n]$ is a function so that $f : V \to [n]$ is a function so that $f(u) \leq f(v)$ if there is an edge directed from u to v in A,*

(see Figure 7 for examples) is equal to

$$(-1)^m \chi_G(-n).$$

Proof. Please check the textbook for a proof that uses the deletion-contraction recurrence. □

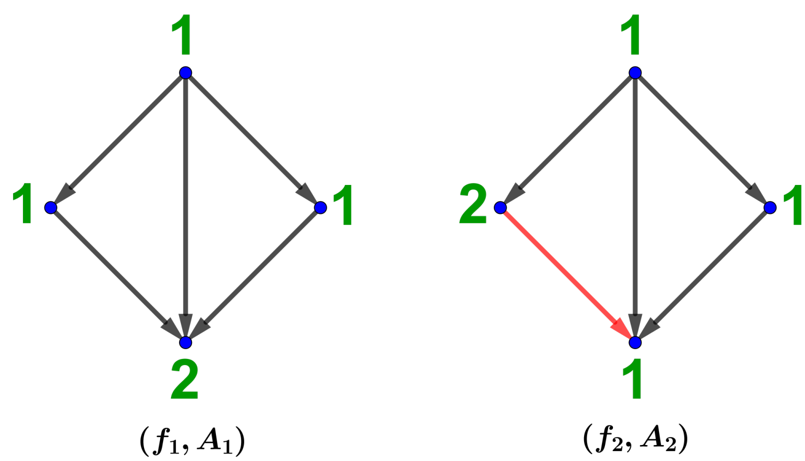

Figure 7: Two pairs of functions and acyclic orientations (f_1, A_1) and (f_2, A_2), where the function is given in green color. The first pair (f_1, A_1) satisfies the condition in Theorem 18 but the second pair (f_2, A_2) since the source of the red vertex (2) is larger than the target of the red vertex (1).

In particular, Theorem 18 implies tha following corollary.

Theorem 19. *Let G be a simple graph with m vertices, and let n be a positive integer. Then*

$$(-1)^m \chi_G(-n) > 0.$$

Proof. By Theorem 18, it suffices to show that there always exists a pair (f, A) of function and acyclic orientation that satisfies the properties in Theorem 18. We can do this by picking an arbitrary acyclic orientation A (we can do this since the graph is simple) and the function $f : V \to [n]$ being the constant function with $f(v) = 1$ for all $v \in V$. This proves the claim. □

Remark 20. Read other parts of Section 5.4 not covered in the notes! △