Math 184

Lecture Notes Section 4.4 *

Instructor: Swee Hong Chan

NOTE: The notes is a summary for materials discussed in the class and is not supposed to substitute the textbook. In particular, the proofs here might omit some details for brevity, and are not supposed to be how you write proofs in the exam. Please refer back to the textbook when studying for exams; materials that appear in the textbook but do not appear in the lecture notes might still be tested. Please send me an email if you find typos.

1 Inversions

We want to study another property of permutations called *inversions*. We continue our tradition of starting with examples before rigorous definitions.

Example 1.
- The permutation 31524 has four inversions. These are

$$(3,1), \quad (3,2), \quad (5,2), \quad \text{and} \quad (5,4).$$

- The permutation 1723456 has five inversions. These are

$$(7,2), \quad (7,3), \quad (7,4), \quad (7,5), \quad (7,6).$$

- The permutation 123456789 has no inversions.
- The permutation 54321 has $\binom{5}{2} = 10$ inversions. These are

$$(5,4), \quad (5,3), \quad (5,2), \quad (5,1), \quad (4,3),$$
$$(4,2), \quad (4,1), \quad (3,2), \quad (3,1), \quad (2,1).$$

- The permutation $n(n-1)\dots 321$ has $\binom{n}{2}$ inversions, where every pair (i,j) with $1 \leq j < i \leq n$ is an inversion of this permutation. △

Definition 2 (Inversions)**.** Let $p = p_1 p_2 \dots p_n$ be a permutation (in the list form). Then we say that the pair (p_i, p_j) is an *inversion* of p if $i < j$ but $p_i > p_j$. The number of inversions will be denoted by $i(p)$. △

Remark 3. Because of the similarity between descents and inversions, it might be worth repeating here the differences between these two notions. We will do so by using the permutation 31524 as an example. This

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permutation has two descents and four inversions, namely

Descents: position 1 and position 3.

Inversions: (3,1), (3,2), (5,2), and (5,4).

- Descents count **consecutive** pairs in the permutation in the wrong order, while inversions count **any** pairs in the permutation in the wrong order.
- Descents are recorded using the **position** of descent in the permutation, while inversions are recorded using the **pair of elements** in the permutation that is in the wrong order.

△

Another way of defining inversions is by using arrow diagrams, drawn during the lecture[1]. One of the immediate consequences of the arrow diagram is the following proposition.

Proposition 4. *For every permutation p, we have $i(p) = i(p^{-1})$.*

Proof. This follows immediately from drawing the arrow diagrams of $i(p)$ and $i(p^{-1})$. The rigorous proof is left to the reader as an exercise. □

Remark 5. Why should we care about inversions? One of the reason is that inversions are used in computing the determinant of a matrix, which is called the *Leibniz formula*[2]: The determinant of the matrix

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix}$$

is equal to

$$\det(A) = \sum_{p \in S_n} (-1)^{i(p)} a_{1,p_1} a_{2,p_2} \dots a_{n,p_n}, \tag{1}$$

where S_n is the set of all permutations of $[n]$. Note that the number of inversions $i(p)$ appears in the formula for determining the sign of the corresponding term in the sum. △

In particular, (1) motivates checking whether $i(p)$ is even or odd. It is actually important enough to get its own name.

Definition 6. A permutation p is called *odd* if $i(p)$ is odd, and is called *even* if $i(p)$ is even. △

This definition comes with an obvious question: How to check if a permutation is even (efficiently)? One of the way to do so is by using permutation matrices.

[1] Read textbook page 206 and 207 if you did not come to the lecture

[2] The interest in this formula is mostly theoretical, as in practise there are faster formulas to compute determinants.

2 Permutation matrices

Again we start with an example before the rigorous definition.

Example 7. Let $p = 3241 = (143)(2)$. The permutation matrix of A_p is

$$A_p = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Let $p = 42315 = (14)(2)(3)(5)$. The permutation matrix of A_p is

$$A_p = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \qquad \triangle$$

Definition 8. Let $p = p_1p_2 \dots p_n$ be a permutation of $[n]$. The *permutation matrix* of p is the $n \times n$ matrix A_p defined by

$$A_p(i,j) := \begin{cases} 1 & \text{if } p_i = j; \\ 0 & \text{otherwise.} \end{cases}$$

$\triangle$

Proposition 9. *Let p be a permutation of $[n]$. Then*

$$\det(A_p) = (-1)^{i(p)}.$$

That is to say, a permutation p is odd if $\det(A_p) = -1$, and is even if $\det(A_p) = 1$.

There exists a fast algorithm for computers to compute $\det(A_p)$, so one can compute this determinant to determine if a permutation is even. *However*, there is actually a simpler way to determine the parity of a permutation, especially when the input is in the cycle notation.

Theorem 10. *Let p be a permutation of $[n]$. Then p is an even permutation if and only if the number of even cycles is even.*

Example 11. We check the permutations that we have seen from Example 1.

- The permutation 31524 is (13542) in the cycle notation. So
  - Number of even cycles $= 0$;
  - Number of inversions $= 4$.

  Indeed, both quantities are even.

- The permutation 1723456 is (1)(276543) in the cycle notation. So
  - Number of even cycles = 1;
  - Number of inversions = 5.

  Indeed, both quantities are odd.
- The permutation 123456789 is (1)(2)(3)(4)(5)(6)(7)(8)(9) in the cycle notation. So
  - Number of even cycles = 0;
  - Number of inversions = 0.

  Indeed, both quantities are even.
- The permutation 54321 is (15)(24)(3) in the cycle notation. So
  - Number of even cycles = 2;
  - Number of inversions = 10.

  Indeed, both quantities are even.

$\triangle$

Remark 12. Note that Theorem 10 merely says that $i(p)$ has the same parity as the number of even cycles of p. What it does NOT say (and it is in fact false) is that $i(p)$ is equal to the number of even cycles. Indeed, this has been observed in the example above. $\triangle$

We now return to the discussion of the permutation matrix A_p. It might appear that our original motivation (computing the parity of $i(p)$) can be done more efficiently *without* using permutation matrices, so why even bother with studying this object? As it turns out, permutation matrix has an (arguably much more) important property that makes it much more useful than just for computing $i(p)$.

Definition 13 (Product of permutations)**.** Let p and q be two permutations of $[n]$, now viewed as bijections from $[n]$ to $[n]$. The product pq of two permutations is defined to be the bijection from $[n]$ to $[n]$ given by

$$pq(i) := p(q(i)) \qquad (i \in [n]).$$

$\triangle$

Example 14. Let $p = 34521$ and $q = 23145$. Then the product pq is the following bijection:

$$pq \text{ maps 1 to } p(q(1)) = p(2) = 4; \tag{2}$$

$$pq \text{ maps 2 to } p(q(2)) = p(3) = 5; \tag{3}$$

$$pq \text{ maps 3 to } p(q(3)) = p(1) = 3; \tag{4}$$

$$pq \text{ maps 4 to } p(q(4)) = p(4) = 2; \tag{5}$$

$$pq \text{ maps 5 to } p(q(5)) = p(5) = 1. \tag{6}$$

Therefore, the permutation pq is 45321 in the list notation. $\triangle$

Theorem 15. *Let p and q be permutations of $[n]$. Then*

$$A_{pq} = A_p A_q.$$

Remark 16. Described in words, representing permutations by the matrix A_p is an operation that preserves the products of two permutations. Described in the language of group theory, the mapping from the group of permutations S_n to the group of $n \times n$ matrices sending p to A_p is a *homomorphism.* Described in the language of representation theory, the matrix representation A_p is the *natural representation* of the group of permutations S_n. Indeed, this *almost trivial* observation of representing permutations as matrices turned out to be an important building block of the representation theory, which has been a very central field in mathematics since its introduction 100 years ago with connections to all areas in mathematics[3]. $\triangle$

Remark 17. Read other examples in Section 4.4 not covered in the notes! $\triangle$

3 Counting inversions

Let $b(n,k)$ be the number of permutations of $[n]$ with k inversions. There exists explicit formula to compute $b(n,k)$, but it is quite complicated to write out. Instead, we have the following result for the generating function $I_n(x)$ of $b(n,k)$'s.

Theorem 18. *For all positive integers $n \geq 2$:*

$$I_n(x) \;=\; \sum_{k=0}^{\binom{n}{2}} b(n,k)x^k \;= (1+x)(1+x+x^2)\ldots(1+x+x^2+\ldots+x^{n-1}). \tag{7}$$

Proof. This theorem can be proved using induction and applying some special recurrence relations. We leave reading the proof of this theorem in the textbook as an exercise. $\square$

Example 19. Let $n = 3$. Note that

- $b(3,0) = 1$ since there is one permutation of $[3]$ with no inversions, namely 123;
- $b(3,1) = 2$, since there are two permutations with 1 inversion, namely 132 and 213;
- $b(3,2) = 2$, since there are two permutations with 2 inversions, namely 231 and 312;
- $b(3,3) = 1$ since there is one permutation of $[3]$ with 3 inversions, namely 321.

This gives us

$$I_n(x) = b(3,0) + b(3,1)x + b(3,2)x^2 + b(3,3)x^3 = 1 + 2x + 2x^2 + x^3.$$

On the other hand, we have

$$(1+x)(1+x+x^2) = (1+x+x^2) + (x+x^2+x^3) = 1 + 2x + 2x^2 + x^3.$$

These two polynomials are indeed equal. $\triangle$

[3] I recommend all students interested in studying pure mathematics to take at least one course on the (classical) representation theory

The right side of (18) is particularly interesting. Macmahon (late 19th century) observed that this polynomial also appears as a generating function of another important series.

Definition 20 (Major index)**.** Let p be a permutation of $[n]$. The *major index* $\text{maj}(p)$ of p is the sum of all descents of p. △

Example 21.

- The permutation 31524 has two descents at position 1 and position 3. So the major index of this permutation is $1 + 3 = 4$.
- The permutation 1743265 has descents at position 2, 3, 4, 6, so the major index of this permutation is $2 + 3 + 4 + 6 = 15$.
- The permutation 12345. . . n has no descent, so the major index of this permutation is 0.
- The permutation $n(n-1)(n-2)\dots 21$ has descent at position 1, 2, 3, $\dots n-1$, so the major index of this permutation is $1 + 2 + 3 + \dots + n - 1 = \frac{n(n-1)}{2}$. △

The major index is named after its discoverer, *Major* Persy Macmahon, who was a real major in the British army[4]. Here is the theorem that he found.
Let $m(n, k)$ be the number of permutations of $[n]$ with major index k.

Theorem 22. *For all positive integers $n \geq 2$,*

$$\sum_{k=0}^{\frac{n(n-1)}{2}} m(n,k)x^k = (1+x)(1+x+x^2)\dots(1+x+x^2+\dots+x^{n-1}). \tag{8}$$

Proof. Please read the proof in the textbook, which uses recursive proofs using generating functions. □

Remark 23. The equality in (18) and (8) implies that $b(n, k) = m(n, k)$ for all n and k. In particular, it suggests that there should be a bijection between permutations with k inversions to permutations with major index k. Such bijection was finally found by Foata and Schützenberger 50 years after Macmahon first proved this result (algebraically). This bijection of Foata in turn has been generalized to many more different settings, as other mathematicians later noticed its connection to (among other things) Standard Young tableau and representation theory of the group of permutations S_n. △

[4]This greatly annoyed Hardy, another great mathematician who was a good friend of Macmahon and was an outspoken pacifist