Math 184

Lecture Notes Section 4.3 *

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NOTE: The notes is a summary for materials discussed in the class and is not supposed to substitute the textbook. In particular, the proofs here might omit some details for brevity, and are not supposed to be how you write proofs in the exam. Please refer back to the textbook when studying for exams; materials that appear in the textbook but do not appear in the lecture notes might still be tested. Please send me an email if you find typos.

1 Cycle structure and exponential generating function

Example 1. Compute the number E_n of permutations of $[n]$ in which every cycle in the permutation has even length, e.g., The cycle

$$(14)\ (2365)\ (78)$$

counts as a permutation in which every cycle has even length, but

$$(14)\ (236)\ (57)$$

does not count, as the cycle (236) has odd length. Also note that

$$(1234)(5)$$

does not count, as the cycle (5) has odd length. $\triangle$

(Partial) answer. Let's try to compute E_n for small values.

- Let $n = 2$. In this case, there is exactly one such permutation, namely

$$(12).$$

- Let $n = 3$. In this case, all the permutations are

$$\begin{array}{lll} (1)(2)(3) & (123) & (132) \\ (12)(3) & (13)(2) & (23)(1). \end{array}$$

  All of them have at least one cycle of odd length! Hence $E_n = 0$ when $n = 3$. We will talk more about this.

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- Let $n = 4$. In this case, $E_n = 9$ and all such permutations are

$$\begin{array}{ccc} (12)(34) & (13)(24) & (14)(23) \\ (1234) & (1243) & (1324) \\ (1342) & (1423) & (1432). \end{array}$$

- Let $n = 5$. In this case, we can actually conclude that $E_n = 0$ even without doing examples. This is because the length n of a permutation is equal to the sum of its cycle lengths. Therefore, a permutation for which all of its cycles have even length must have n=even.

- Let $n = 6$. There are too many such permutations so listing all of them will not be practical. However, one can use computers to calculate that $E_6 = 225$.

Here are the values for E_n for larger values of n:

$$E_8 = 11025; \qquad E_{10} = 893025, \qquad E_{12} = 108056025, \qquad E_{14} = 18261468225.$$

One can then use either divine inspiration or Online Encyclopedia of Integer Sequences to conjecture that E_{2m} is given by the formula

$$E_{2m} = (1 \cdot 3 \cdot 5 \cdots (2m-1))^2 = (2m-1)!!^2. \tag{1}$$

□

We will now learn how to use generating function to prove (1). This technique can be summarized as the following theorem.

Theorem 2. *Let C be any set of positive integers, and let $g_C(n)$ be the number of permutations of length n whose cycle lengths are all elements of C. Then*

$$G_C(x) = \sum_{n=0}^{\infty} g_C(n) \frac{x^n}{n!} = \exp\left(\sum_{n \in C} \frac{x^n}{n} \right).$$

Proof. The proof uses the compositional formula for the exponential generating function that we learned back in Section 3. We will omit the proof from the lecture notes as the application of the theorem is the more interesting part of the story here. Please read the textbook for a rigorous proof. □

We will now see some applications of Theorem 2.

Example 3. Let $C = \{1, 2, 3, \ldots\}$ be the set of all positive integers. Show that $g_C(n) = n!$. △

Proof. Note that $g_C(n)$ counts the number of permutations of $[n]$ for which its cycles all have positive lengths. This is literally just counting all permutations of $[n]$, which has $n!$. This is great, but let's try if we can derive the same thing using Theorem 2.

It follows from Theorem 2 that

$$\begin{aligned}
G_C(x) &= \exp\left(\sum_{n\in C}\frac{x^n}{n}\right)\\
&= \exp\left(\sum_{n=1}^{\infty}\frac{x^n}{n}\right)\\
&= \exp(-\ln(1-x))\\
&= \exp\left(\ln\frac{1}{1-x})\right)\\
&= \frac{1}{1-x}\\
&= \sum_{n=0}^{\infty}x^n\\
&= \sum_{n=0}^{\infty}n!\frac{x^n}{n!}.
\end{aligned}$$

It then follows that $g_c(n) = n!$, as desired. □

Example 4. Let $C = \{2, 4, 6, \ldots\}$ be the set of all even positive integers. Show that

$$g_C(n) = E_n = \begin{cases} 0 & \text{if } n = 2m+1 \text{ (odd);} \\ (1\cdot 3\cdot 5\cdots(2m-1))^2 \ = (2m-1)!!^2 & \text{if } n = 2m \text{ (even).} \end{cases} \qquad \triangle$$

Proof. It follows from Theorem 2 that

$$\begin{aligned}
G_C(x) &= \exp\left(\sum_{n\in C}\frac{x^n}{n}\right)\\
&= \exp\left(\sum_{n=2m}\frac{x^n}{n}\right)\\
&= \exp\left(\sum_{m=1}\frac{x^{2m}}{2m}\right)\\
&= \exp\left(\frac{1}{2}\sum_{m=1}\frac{(x^2)^m}{m}\right)\\
&= \exp\left(\frac{1}{2}(-\ln(1-x^2))\right)\\
&= \exp\left(\frac{1}{2}\ln\frac{1}{1-x^2}\right)\\
&= \exp\left(\ln\sqrt{\frac{1}{1-x^2}}\right)\\
&= \sqrt{\frac{1}{1-x^2}}.
\end{aligned}$$

We will now use the binomial series theorem to compute the power series of $\sqrt{\frac{1}{1-x^2}}$.

Recall that the normal binomial theorem for integer n is

$$(1+z)^n = \sum_{k=0}^{n} \binom{n}{k} z^k.$$

Another way to write the series above is

$$(1+z)^n = \sum_{k=0}^{\infty} \binom{n}{k} z^k,$$

since $\binom{n}{k} = 0$ if $k > n$ and n is an integer. Now recall that the binomial coefficient can be written as

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)(n-2)\dots(n-k+1)}{k!}.$$

Note that the right side of the equation above makes sense even when n is not an integer anymore.

Theorem 5 (Binomial series theorem)**.** *Let z be a variable, and let r be a real number. Then the power series of $(1+z)^r$ is equal to*

$$(1+z)^r = \sum_{k=0}^{\infty} \frac{r(r-1)(r-2)\dots(r-k+1)}{k!} z^r. \qquad \square$$

Applying this to our original problem, we have

$$\sqrt{\frac{1}{1-x^2}} \;=\; (1-x^2)^{-\frac{1}{2}}.$$

Substituting $z = -x^2$ and $r = -\frac{1}{2}$ and $m = k$, we get

$$\begin{aligned}
\sqrt{\frac{1}{1-x^2}} \;&=\; \sum_{m=0}^{\infty} \frac{(-\frac{1}{2})(-\frac{3}{2})\dots(-\frac{2m-1}{2})}{m!}(-x^2)^m \\
&= \sum_{m=0}^{\infty} \frac{(\frac{1}{2})(\frac{3}{2})\dots(\frac{2m-1}{2})}{m!} x^{2m} \\
&= \sum_{m=0}^{\infty} \frac{(2m-1)!!}{m!2^m} x^{2m} \\
&= \sum_{m=0}^{\infty} \frac{(2m-1)!!(2m)!}{m!2^m} \frac{x^{2m}}{(2m)!}.
\end{aligned}$$

It then follows that $g_C(n) = 0$ if $n = 2m+1$ and $g_C(n) = \frac{(2m-1)!!(2m)!}{m!2^m}$ if $n = 2m$. In order to get to the form we see in the beginning, note that

$$\begin{aligned}
g_C(2m) =&(2m-1)!!\frac{(2m)!}{m!2^m} \\
=&(2m-1)!!\frac{1\cdot 2\cdot 3\cdot 4\cdots(2m-1)\cdot(2m)}{(1\cdot 2\cdot 3\cdots m)2^m} \\
=&(2m-1)!!\frac{1\cdot 2\cdot 3\cdot 4\cdots(2m-1)\cdot(2m)}{((1\cdot 2)\cdot(2\cdot 2)\cdot(3\cdot 2)\cdots(m\cdot 2)} \\
=&(2m-1)!!\frac{1\cdot 2\cdot 3\cdot 4\cdots(2m-1)\cdot(2m)}{2\cdot 4\cdot 6\cdots 2m} \\
=&(2m-1)!!\frac{1\cdot \cancel{2}\cdot 3\cdot \cancel{4}\cdots(2m-1)\cdot\cancel{(2m)}}{\cancel{2}\cdot\cancel{4}\cdot\cancel{6}\cdots\cancel{(2m)}} \\
=&((2m-1)!!)^2.
\end{aligned}$$

Exercise 6. Read Example 4.36 from the textbook. $\triangle$

Remark 7. Read other examples in Section 4.3 not covered in the notes! $\triangle$