Math 184

Lecture Notes Section 4.2 *

Instructor: Swee Hong Chan

NOTE: The notes is a summary for materials discussed in the class and is not supposed to substitute the textbook. In particular, the proofs here might omit some details for brevity, and are not supposed to be how you write proofs in the exam. Please refer back to the textbook when studying for exams; materials that appear in the textbook but do not appear in the lecture notes might still be tested. Please send me an email if you find typos.

1 Permutation (as a bijection)

In Section 4.1 we define permutation as an ordering of elements in the set $\{1, 2, \dots, n\}$. We will now look at permutations from a different point of view.

Definition 1 (Permutation as a bijection)**.** A *permutation* is a bijection mapping the set $[n]$ into $[n]$. $\triangle$

Example 2. The following are all the permutations on [3], both as lists and as bijections:

List	*Functions*
123	$f(1) = 1;\quad f(2) = 2;\quad f(3) = 3$
132	$f(1) = 1;\quad f(2) = 3;\quad f(3) = 2$
213	$f(1) = 2;\quad f(2) = 1;\quad f(3) = 3$
231	$f(1) = 2;\quad f(2) = 3;\quad f(3) = 1$
312	$f(1) = 3;\quad f(2) = 1;\quad f(3) = 2$
321	$f(1) = 3;\quad f(2) = 2;\quad f(3) = 1.$

$\triangle$

Writing permutations as a bijection can be slightly time-consuming if we keep the notation used above. Therefore we will write them using *cycle notation*, and we keep our tradition of presenting examples before introducing definitions.

*Version date: Friday 21^{st} February, 2020, 11:21

Example 3. The following are all the permutations on [3], both as lists and as bijections:

List	Functions	Cycle notations
123	$f(1)=1;\quad f(2)=2;\quad f(3)=3$	(1)(2)(3)
132	$f(1)=1;\quad f(2)=3;\quad f(3)=2$	(1)(23)
213	$f(1)=2;\quad f(2)=1;\quad f(3)=3$	(12)(3)
231	$f(1)=2;\quad f(2)=3;\quad f(3)=1$	(123)
312	$f(1)=3;\quad f(2)=1;\quad f(3)=2$	(132)
321	$f(1)=3;\quad f(2)=2;\quad f(3)=1$	(13)(2).

△

Definition 4 (Cyclic notations)**.** Let $f : [n] \to [n]$ be a permutation. In cycle notation, the elements of $[n]$ are put inside a number of parentheses, i.e.,

$$(a_1 a_2 \ldots a_k)\ (b_1 b_2 \ldots b_\ell)\ \cdots\ (z_1 z_2 \ldots z_h),$$

ordered so that $f(j)$ immediately follows j or, if j is the last listed element of the cycle, then $f(j)$ is the first element of the cycle. △

Example 5. Here are examples of permutations with corresponding cyclic notations:

- The permutation 692174583 corresponds to

  $$(164)\ (293)\ (57)\ (8).$$

  Note that each cycle is equivalent up to a rotation. That is to say,

  $$(164)\ (293)\ (57)\ (8) \quad \text{is equivalent to} \quad (641)\ (293)\ (57)\ (8),$$

  since (164) and (641) are two cycles that are equivalent up to rotation. However

  $$(164)\ (293)\ (57)\ (8) \quad \text{is not equivalent to} \quad (146)\ (293)\ (57)\ (8),$$

  since the cycles (164) and (146) are different.

  Note that the order **among** the cycles are irrelevant, as the following cyclic notation corresponds to the same permutation:

  $$(293)\ (164)\ (57)\ (8).$$

  However, the order inside the cycles are very relevant, as the following cyclic notation corresponds to a different permutation:

  $$(239)\ (164)\ (57)\ (8),$$

  corresponds to the list 639174582.

- The permutation 123456789 corresponds to

$$(1)\ (2)\ (3)\ (4)\ (5)\ (6)\ (7)\ (8)\ (9).$$

- The permutation 987654321 corresponds to

$$(19)\ (28)\ (37)\ (46)\ (5).$$

- The permutation 87654321 corresponds to

$$(18)\ (27)\ (36)\ (45).$$

△

Exercise 6. Read the textbook for a proof that the cyclic notation is unique to each permutation, up to swapping of the cycles. △

2 Stirling number of the first kind

Definition 7. The *Stirling number of the first kind* $c(n, k)$ is the number of permutations of $[n]$ having exactly k cycles in its cyclic notation. △

Example 8. Consider $n = 3$.

- The Stirling number $c(3, 0)$ is equal to 0, as all permutations has at least one cycle!
- The Stirling number $c(3, 1)$ is equal to 2, as there are two permutations with exactly $k = 1$ cycle, namely

$$(123) \quad \text{and} \quad (132).$$

- The Stirling number $c(3, 2)$ is equal to 3, as there are three permutations with exactly $k = 2$ cycles, namely

$$(12)\ (3); \qquad (13)\ (2); \qquad (23)\ (1).$$

- The Stirling number $c(3, 3)$ is equal to 1, as there is one permutation with exactly $k = 3$ cycles, namely

$$(1)(2)(3).$$

△

Another example can be seen in the picture in the lecture.

Remark 9. Since we are talking about Stirling number of the first kind, it is natural to refresh our minds regarding Stirling number of the second kind, which we learn in Section 2. Recall that Stirling number of the second kind $\left\{ {n \atop k} \right\}$ is the number of ways to partition $[n]$ into k non-empty subsets. Here are all such partitions for $n = 4$ and $k = 2$:

$$\{1\}, \{2, 3, 4\}; \qquad \{2\}, \{1, 3, 4\}; \qquad \{3\}, \{1, 2, 4\}; \qquad \{4\}, \{1, 2, 3\},$$
$$\{1, 2\}, \{3, 4\}; \qquad \{1, 3\}, \{2, 4\}; \qquad \{1, 4\}, \{2, 3\}.$$

△

Remark 10 (Historical)**.** Stirling number of the first and second kind was first studied by James Stirling in 18th century. Interestingly, Stirling motivation was algebraic and the combinatorial aspect of Stirling number was probably unknown to Stirling himself. Indeed, the combinatorial meaning of Stirling number only appeared in the literature during the early 20-th century (Whitworth 1901, Nielsen 1904) to the best of the instructor's knowledge.

So why was Stirling interested in Stirling number? Recall from our previous discussion on Stirling number of the second kind that the mathematicians of that era were interested in the art of writing "nice" functions as power series (for which Taylor's series in Calculus being one of the crystallization of this pursuit). In the process of pursuing research on this direction Stirling noticed these two interesting polynomial identities.

Denote by $(x)_n$ the polynomial in x of degree n given by

$$(x)_n := x(x-1)(x-2)\dots(x-n+1).$$

This polynomial is known as the *falling factorial* of n.

The first identity that Stirling noticed is

$$(x)_n = \sum_{k=0}^{n}(-1)^{n-k}c(n,k)x^k = (-1)^n c(n,0) + (-1)^{n-1}c(n,1)x + (-1)^{n-2}c(n,2)x^2 + \dots + c(n,n)x^n. \quad (1)$$

We will prove this identity later in this lecture notes.

The second identity that Stirling noticed is

$$x^n = \sum_{k=0}^{n}\left\{{n \atop k}\right\}(x)_k = \left\{{n \atop 0}\right\} + \left\{{n \atop 1}\right\}(x)_1 + \left\{{n \atop 2}\right\}(x)_2 + \dots + \left\{{n \atop n}\right\}(x)_n. \quad (2)$$

Philosophically, Stirling number of the first kind transforms monomials x^n into falling factorials $(x)_n$, while Stirling number of the second kind transforms falling factorials $(x)_n$ back into monomials x^n, and one can argue that Stirling number of the first kind is dual to the second kind. Studying objects with this flavor of duality property is a separate (sub)field in combinatorics[1]. △

3 Recurrence relation

One signature property of Stirling number is that it satisfies a variation of recurrence relation seen with binomial coefficients. In the case of Stirling number of the first kind, the relation takes the following form.

Theorem 11. *Let n be a positive integer, and let k be a nonnegative integer such that $k \le n$. Then*

$$c(n,k) = c(n-1,k-1) + (n-1)c(n-1,k).$$

Proof. We leave the proof as an exercise for the reader, as the proof uses same ideas that we have encountered in the proof of the recurrence relation of binomial coefficients and Stirling numbers of the second kind. □

[1] Ask the instructor for more details if you are interested.

Remark 12 (Historical)**.** This is a good time to recall the recurrence relations of binomial coefficients and Stirling numbers of the second kind, namely

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k};$$
$$\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k \left\{ {n-1 \atop k} \right\}.$$

Note that there exist Stirling numbers of the third kind $L(n, k)$, which satisfies the following recurrence relation:

$$L(n, k) = L(n-1, k-1) + (n+k-1)L(n-1, k).$$

Interestingly, this number is not due to Stirling! It was instead discovered by Lah in 1954, and therefore this number is often called *Lah numbers*. Notably, Stirling number of the third kind is the unique one among the three kinds to have a nice closed form, namely

$$L(n, k) = \binom{n-1}{k-1} \frac{n!}{k!}.$$ △

4 Combinatorial identities

Here are some simple combinatorial identities of Stirling numbers of the first kind:

- We have $c(n, 0) = 0$ for $n \geq 1$, since any permutation has at least one cycle.
- We have $c(n, n) = 1$ for $n \geq 1$, since the only permutation with n cycles in its cycle notation is the identity permutation $(1)(2)(3) \ldots (n)$.
- We have $c(n, 1) = (n-1)!$. This is because permutations with exactly one cycle in cycle notation has the form
$$(a_1 a_2 \ldots a_n).$$
This is exactly counting the number of seating arrangement of n people at a circular table, which we have known is equal to $(n-1)!$.
- The sum of Stirling number of the first kind is equal to
$$n! = \sum_{k=1}^{n} c(n, k) = c(n, 0) + c(n, 1) + \ldots + c(n, n).$$
Indeed, the right side of the equation above counts the number of permutations of $[n]$ with any number of cycles, which is equal to $n!$.

We now prove Stirling's identity (2) promised previously.

Theorem 13. *We have for all $n \geq 1$ that*

$$(x+n-1)(x+n-2) \ldots (x+2)(x+1)x = \sum_{k=0}^{n} c(n, k) x^k.$$

Remark 14. Note that the textbook wrote $\sum_{k=1}^{n} c(n, k) x^k$ instead of $\sum_{k=0}^{n} c(n, k) x^k$ for the sum in the right side of the equation above (the indexing of k starts from 1 rather than 0). Note that the two expressions are equal (since $c(n, 0) = 0$), so there are no contradictions! △

Proof of Theorem 13. We are going to prove the claim by induction on n. The base case $n = 1$ can be checked by

$$(x+n-1)\dots(x+1)x = x = c(1,0) + c(1,1)x = \sum_{k=0}^{n} c(n,k)x^k.$$

Suppose that the claim is true for $n-1$, we now prove the claim for n. We have

$$\begin{aligned}
&(x+n-1)(x+n-2)(x+n-3)\dots(x+2)(x+1)x \\
&= (x+n-1)\sum_{k=0}^{n-1} c(n-1,k)x^k \qquad \text{(by induction)} \\
&= x\sum_{k=0}^{n-1} c(n-1,k)x^k + (n-1)\sum_{k=0}^{n-1} c(n-1,k)x^k \\
&= \sum_{k=0}^{n-1} c(n-1,k)x^{k+1} + \sum_{k=0}^{n-1}(n-1)c(n-1,k)x^k \\
&= \sum_{\ell=1}^{n} c(n-1,\ell-1)x^\ell + \sum_{k=0}^{n-1}(n-1)c(n-1,k)x^k \qquad \text{(Substitute } \ell = k+1\text{)} \\
&= \sum_{\ell=0}^{n} c(n-1,\ell-1)x^\ell + \sum_{k=0}^{n}(n-1)c(n-1,k)x^k \qquad \text{(Since } c(n-1,-1) = 0 \text{ and } c(n-1,n) = 0\text{)} \\
&= \sum_{k=0}^{n} c(n-1,k-1)x^k + \sum_{k=0}^{n}(n-1)c(n-1,k)x^k \qquad \text{(Substitute } k = \ell\text{)} \\
&= \sum_{k=0}^{n}(c(n-1,k-1) + (n-1)c(n-1,k))x^k \qquad \text{(Substitute } k = \ell\text{)} \\
&= \sum_{k=0}^{n} c(n,k)x^k \qquad \text{(By the recurrence relation),}
\end{aligned}$$

as desired. □

Remark 15. Being a class in enumerative combinatorics, it might seem strange that we do not try to count the number $c(n,k)$ at all! This is because no simple closed-form formula exists for $c(n,k)$! This should not be too surprising as most of the interesting mathematical objects cannot be described by simple formulas[2]. With that being said, one might still wonder, if we ask the **approximate value** of $c(n,k)$ rather than the **exact value**, then it might be possible to get a simple formula after all. Indeed, for "very large" n and "very small" k, we have

$$c(n,k) \;\approx\; \frac{(n-1)!}{(k-1)!}(\gamma + \ln(n-1))^{k-1},$$

where $\gamma = 0.57721\dots$ is the Euler-Mascheroni constant. The field of deriving the **approximate count** rather than the **exact count** of combinatorials object is called *asymptotic combinatorics*[3]. △

[2]One might argue that is what makes it interesting in the first place

[3]This field is in fact one of the instructor's research interest.

5 Permutations of a given cycle type

A cycle type of a permutation $[n]$ is a vector $(a_1, \ldots, a_n)$ of nonnegative integers that is to be defined. We continue our tradition of starting with examples rather than rigorous definitions.

Example 16. Let $n = 3$. The cycle type of permutations of $[n]$ is given by

$$\begin{array}{lll} (1)(2)(3) & \text{has cycle type} & (3,0,0) \\ (12)(3) & \text{has cycle type} & (1,1,0) \\ (13)(2) & \text{has cycle type} & (1,1,0) \\ (23)(1) & \text{has cycle type} & (1,1,0) \\ (123) & \text{has cycle type} & (0,0,1) \\ (132) & \text{has cycle type} & (0,0,1). \end{array}$$

△

Definition 17. The cycle type $(a_1, \ldots, a_n)$ of a permutation p of $[n]$ is given by

$$a_i := \text{number of cycles of } p \text{ with length } i.$$

△

Example 18. Here are some more examples:

- (12) (345) (6) (789) has cycle type $(1, 1, 2, 0, 0, \ldots, 0)$.
- (123456789) has cycle type $(0, 0, \ldots, 0, 1)$.
- (1)(2)(3)(4)(5)(6)(7)(8)(9) has cycle type $(9, 0, 0, \ldots, 0)$. △

One of the properties of a cycle type $(a_1, \ldots, a_n)$ is that

$$a_1 + 2a_2 + 3a_3 + \ldots + na_n = n.$$

Indeed, this is because the left side of the equation above counts the number of elements in the permutation! Perhaps surprisingly, there is a nice closed-form formula for number of permutations of any given cycle type (even though such a formula doesn't exist for Stirling number!), as shown by the next theorem.

Theorem 19. *Let n be any positive integer, and let $(a_1, a_2, \ldots, a_n)$ be a vector of nonnegative integers such that $\sum_{i=1}^n ia_i = n$. Then the number of permutations of $[n]$ of type $(a_1, a_2, \ldots, a_n)$ is equal to*

$$\frac{n!}{1^{a_1} 2^{a_2} \ldots n^{a_n} a_1! a_2! \ldots a_n!}.$$

Proof. We left the proof as an exercise to the reader. Note that it can be solved by combining ideas such as the product principle and the division principle, which we should be very familiar with by now. □

Remark 20. Read other examples in Section 4.2 not covered in the notes! △