Math 184

Lecture Notes Section 4.1 *

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NOTE: The notes is a summary for materials discussed in the class and is not supposed to substitute the textbook. In particular, the proofs here might omit some details for brevity, and are not supposed to be how you write proofs in the exam. Please refer back to the textbook when studying for exams; materials that appear in the textbook but do not appear in the lecture notes might still be tested. Please send me an email if you find typos.

1 Permutation (recap)

We have learned about permutations from Section 1, but that has been a long time ago, and we have 3 different equivalent definitions, so it might be a good idea to recap the definition.

Definition 1 (Permutation)**.** A *permutation* of $[n]$ is just a listing of 1,2,3,..., n in a particular order. △

Here are some examples:

- The following are permutations of [3]:

$$123 \quad 132 \quad 213 \quad 231 \quad 312 \quad 321.$$

- The following are permutations of [4]:

$$\begin{array}{cccccc} 1234 & 1243 & 1324 & 1342 & 1423 & 1432 \\ 2134 & 2143 & 2314 & 2341 & 2413 & 2431 \\ 3124 & 3142 & 3214 & 3241 & 3412 & 3421 \\ 4123 & 4132 & 4213 & 4231 & 4312 & 4321. \end{array}$$

The number of permutations of $[n]$ is equal to $n!$. Indeed, we say that in the example above when $n = 3$ (in which we have 6 permutations) and $n = 4$ (in which we have 24 permutations).

2 Ascents and descents

We are interested in studying the ascending runs of a permutation. From my personal experience, the best way to understand ascents and descents is to try out several examples first *before* seeing the actual definition.

Example 2. • The permutation 245169378 has two descents:

$$245 \blacksquare 169 \blacksquare 378$$

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- The permutation 983576412 has five descents:

$$9 \blacksquare 8 \blacksquare 357 \blacksquare 6 \blacksquare 34 \blacksquare 12$$

- The permutation 123456789 has no descents:

$$123456789$$

- The permutation 987654321 has eight descents:

$$9 \blacksquare 8 \blacksquare 7 \blacksquare 6 \blacksquare 5 \blacksquare 4 \blacksquare 3 \blacksquare 2 \blacksquare 1$$

△

Definition 3 (Ascent and descent)**.** Let $p = p_1 \dots p_n$ be a permutation. We say that $i \in \{1, \dots, n-1\}$ is a *descent* if $p_i > p_{i+1}$, and is called an *ascent* if $p_i < p_{i+1}$. △

Remark 4. Note that the descents are the *positions* of p, not its entries! △

Example 5.

- The permutation 245169378 has two descents at position 3 and 6 :

$$245 \blacksquare 169 \blacksquare 378$$

- The permutation 983576412 has five descents at position 1,2,5,6,7:

$$9 \blacksquare 8 \blacksquare 357 \blacksquare 6 \blacksquare 4 \blacksquare 12$$

- The permutation 123456789 has no descents:

$$123456789$$

- The permutation 987654321 has eight descents at position 1,2,3,4,5,6,7,8:

$$9 \blacksquare 8 \blacksquare 7 \blacksquare 6 \blacksquare 5 \blacksquare 4 \blacksquare 3 \blacksquare 2 \blacksquare 1$$

△

Definition 6 (Ascending runs)**.** An *ascending run* of a permutation is a block that is bordered by black squares $\blacksquare$. △

Example 7.

- The permutation 245169378 has three ascending runs:

$$245 \blacksquare 169 \blacksquare 378$$

- The permutation 983576412 has six ascending runs:

$$9 \blacksquare 8 \blacksquare 357 \blacksquare 6 \blacksquare 34 \blacksquare 12$$

- The permutation 123456789 has one ascending run:

$$123456789$$

- The permutation 987654321 has nine ascending runs:

$$9 \blacksquare 8 \blacksquare 7 \blacksquare 6 \blacksquare 5 \blacksquare 4 \blacksquare 3 \blacksquare 2 \blacksquare 1 \qquad \triangle$$

Remark 8. See the textbook for a rigorous (non-intuitive) definition of ascending runs. $\triangle$

Exercise 9. Show that the number of ascending runs is always one more than the number of descents. $\triangle$

3 Eulerian numbers

Definition 10 (Eulerian numbers)**.** The *Eulerian number* $A(n,k)$ is the number of permutations of $[n]$ with exactly k ascending runs. $\triangle$

Example 11.
- Let $n = 3$ and $k = 2$. In this case, $A(n,k) = 4$, as there are four permutations of $[3]$ with exactly 2 ascending runs, namely:

$$132; \qquad 213; \qquad 231; \qquad 312.$$

Indeed, we can see it by

$$13 \blacksquare 2; \qquad 2 \blacksquare 13; \qquad 23 \blacksquare 1; \qquad 3 \blacksquare 12.$$

- We have $A(n,1) = 1$, as there is exactly one permutation of $[n]$ with one ascending run, namely:

$$1234 \ldots n.$$

- We have $A(n,n) = 1$, as there is exactly one permutation of $[n]$ with n ascending runs, namely:

$$n(n-1)(n-2)\ldots 1. \qquad \triangle$$

Remark 12 (Historical notes)**.** Eulerian numbers were named after Leonhard Euler, who were studying these numbers as coefficients of the Euler polynomial. He was interested in using Eulerian polynomial to compute the values of the *Dirichlet eta function* at negative integers, namely

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \ldots \qquad \text{for } s = -1, -2, -3, -4, \ldots$$

Note that the reason why Eulerian numbers can be used to compute $\eta(s)$ is not obvious at all, and this story will be told in future lecture notes.

It is very hard to resist the temptation of discussing Riemann zeta function after introducing eta function. The *Riemann zeta function* is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \ldots \qquad \text{for } s \in \mathbb{C} - \{1\}.$$

Note that the definition above makes sense only when the real part of s is greater than 1, but there is another way to define $\zeta(s)$ that works for all complex numbers not equal to 1.

Perhaps the biggest open problem in mathematics is to determine the zeros of the zeta function:

Problem 13 (Riemann hypothesis)**.** If $\zeta(s) = 0$, then either

- $s = -2, -4, -6, -8, \ldots$; or
- $s = \frac{1}{2} + ti$, for some real t. That is to say, the real part of s is $\frac{1}{2}$. △

The instructor promises to give 10 extra credits for students who manage to solve Riemann hypothesis. △

As you might have expected from a class on enumerative combinatorics, we want to count the number $A(n, k)$.

Theorem 14. *For all nonnegative integers n and k satisfying $k \leq n$, we have*

$$A(n, k) = \sum_{i=0}^{k} (-1)^i \binom{n+1}{i} (k - i)^n.$$

Let's check the formula for small values of k to build intuition.

- Let's try out $k = 0$. In this case we know that $A(n, k) = 0$ since all permutations of $[n]$ has at least one ascending run. Indeed, the formula tells us that

$$A(n, 0) = \sum_{i=0}^{0} (-1)^i \binom{n+1}{i} (0 - i)^n = (-1)^0 \binom{n+1}{0} (0 - 0)^n = 0.$$

- Let's try out $k = 1$. In this case we know that $A(n, k) = 1$ since $123 \ldots n$ is the only permutation with 1 ascending run. Indeed, the formula tells us that

$$A(n, 1) = \sum_{i=0}^{1} (-1)^i \binom{n+1}{i} (1 - i)^n \;=\; (-1)^0 \binom{n+1}{0} (1 - 0)^n + (-1)^1 \binom{n+1}{1} (1 - 1)^n \;=\; 1.$$

- The case $k = 2$ is drastically more complicated than the previous two examples. I recommend reading the textbook for an analysis of this case.

We now present the proof of Theorem 14.

Proof of Theorem 14. We will start with counting objects that we call an *arrangement*.

An n, k*-arrangement* of a permutation of $[n]$ with k blocks such that each block is increasing. Here are some examples to build intuition:

- The following is an n, k-arrangement with $n = 9$ and $k = 4$:

  245 ■ 136 ■ 79 ■ 8

  Note that this permutation has only three ascending runs, as the black square is removable.

- The following is an n, k-arrangement with $n = 5$ and $k = 5$:

  ■ 124 ■ ■ 35 ■

  Note that in this example we have 5 blocks, but the first, third, and fifth block is empty. In particular, note that in this example there is a box at position 0 (i.e., before the first number) and a box at position n.

- The following is not an n, k-arrangement ($n = 8$, $k = 3$):

$$1475 \blacksquare 23 \blacksquare 68.$$

  This example is not an arrangement because the block 1475 is not increasing.

We now claim that

The number of n, k-arrangements is equal to k^n.

This can be seen from the following way to construct n, k-arrangements.

- Start with setting up $k - 1$ black squares to form k empty blocks, e.g.

$$\blacksquare \quad\quad \blacksquare \quad\quad \blacksquare$$

  Note that there are 4 empty blocks in the example above constructed by using 3 squares.

- Now, for each number i, randomly put it into any of the k blocks, e.g.,

$$1457 \blacksquare 23 \blacksquare 69 \blacksquare 8.$$

  Note that each number has k choices. Remember to write the numbers in each block in increasing order.

- Voila! we have constructed an n, k-arrangement. It follows from the product principle that there are exactly k^n many such arrangements.

Note that in an n, k-arrangement, the number of ascending runs can be smaller than k, e.g.,

$$245 \blacksquare 136 \blacksquare \; \blacksquare 79 \blacksquare 8$$

This permutation has only three ascending runs despite having 5 blocks. This is because the two squares between 136 and 79 can be removed and the resulting block is still increasing, i.e.,

$$245 \blacksquare 13679 \blacksquare 8.$$

It then follows that an n, k-arrangement corresponds to a permutation with k ascending runs if and only if none of the boxes are removable! That is to say,

$$A(n, k) = \text{number of } n, k\text{-arrangements with no removable boxes.}$$

Here are some examples:

$$1457 \blacksquare 2369 \blacksquare 8$$

$$245 \blacksquare 13679 \blacksquare 8.$$

We want to count $A(n, k)$ by using inclusion-exclusion principle. It is therefore natural that we should count the set

A_S:= the number of n,k-arrangements with removable squares in position (non-empty) $S \subseteq \{0,1,\ldots,n\}$.

Here are some examples:

- Here is an arrangement with removable squares (colored in red) in position 2 and 6.

  13 ■ 46 ■ 25 ■ 78

- Here is an arrangement with removable squares (colored in red) in position 0 and 3.

  ■ 123 ■ ■ 45678

  Note that here the box at position 0 is (always) removable. Also note that there are two boxes in position 3, so one of them is removable. In this situation, we adopt the convention that the rightmost one is the one that is removable.

- Here is an arrangement with removable squares (colored in red) in position 2 and 6.

  12 ■ 36 ■ 45 ■ 78

  Note that this arrangement is still counted by A_S when $S = \{2\}$! That is to say, it is fine if A_S counts arrangement with more removable squares than required.

It follows from the exclusion-inclusion principle that

$$A(n,k) = \#\text{ of } n,k\text{-arrangements} - \sum_{\substack{S\subseteq\{0,1,\ldots,n\}\\ 1\le|S|\le k-1}} (-1)^{|S|+1} A_S.$$

Note that here S ranges only on subsets with size between 1 and $k-1$, since there can be at most $k-1$ squares!

We now count the the number A_S of n,k-arrangements with removable squares in position S. We claim that

$$\text{The number of } n,k\text{-arrangements is equal to } (k-|S|)^n.$$

This can be seen from this method of constructing such arrangements:

- Let $|S| = i$. Start with setting up $k-i-1$ black squares to form $k-i$ empty blocks, e.g., let $n = 9$, $k = 7$ and $i = 3$,

  ■ ■ ■

  Note that there are $k-i = 4$ empty blocks in the example above formed by using $k-i-1 = 3$ black squares.

- Now, for each number in $\{1,\ldots,n\}$, randomly put it into any of the $k-i$ blocks, e.g.,

  1457 ■ 23 ■ 69 ■ 8.

  Note that each number has $k-i$ choices. Remember to write the numbers in each block in increasing order.

- Now, using the remaining i red squares, put each square in the position instructed by S, e.g., $S = \{0, 4, 7\}$,

$$\textcolor{red}{\blacksquare}\ 1457\ \blacksquare\ \textcolor{red}{\blacksquare}\ 23\ \blacksquare\ 6\ \textcolor{red}{\blacksquare}\ 9\ \blacksquare\ 8.$$

  Note that, when there are already black squares in the prescribed position (such as in position 4 above), the new red square will be put into the rightmost location. Also note that, the red squares we put in this way will always be removable.

- Voila! we have constructed all n, k-arrangements with removable square boxes in S. It follows from the product principle that there are exactly $(k-i)^n$ many such arrangements.

We now substitute the value of A_S into the inclusion-exclusion principle to get

$$\begin{aligned}
A(n,k) =& \# \text{ of } n,k\text{-arrangements} - \sum_{\text{nonempty } S \subseteq \{0,1,\ldots,n\}} (-1)^{|S|+1} A_S \\
=& k^n - \sum_{\substack{S \subseteq \{0,1,\ldots,n\} \\ 1 \le |S| \le k-1}} (-1)^{|S|+1} (k - |S|)^n \\
=& k^n - \sum_{\substack{S \subseteq \{0,1,\ldots,n\} \\ 1 \le |S| \le k-1}} (-1)^{i+1} (k-i)^n \qquad (\text{substitute } i = |S|) \\
=& k^n - \sum_{i=1}^{k-1} (-1)^{i+1} \binom{n+1}{i} (k-i)^n \\
=& \sum_{i=0}^{k} (-1)^i \binom{n+1}{i} (k-i)^n.
\end{aligned}$$

This completes our proof. □

4 Triangular recurrence for Eulerian numbers

Just like Stirling numbers and and binomial coefficient, Eulerian numbers also satisfy a triangular recurrence.

Theorem 15. *Let k and n be nonnegative integers satisfying $k \le n$. Then*

$$A(n,k) = kA(n-1,k) + (n-k+1)A(n-1,k-1).$$

Proof. We will prove the theorem by bijective argument. Note that $A(n,k)$ counts the number of permutations of $[n]$ with k ascending runs, or equivalently, $k-1$ descents.

Let p be a permutation counted by $A(n,k)$. Let p' be the permutation obtained by omitting the entry n. There are two possibilities:

- When the omission of n leaves the number of descents unchanged. In this case, the resulting permutation is a permutation of $[n-1]$ with $k-1$ many descents, so there are $A(n-1,k)$ many of them. Now note that, to reconstruct the permutation p from p', there are two different scenarios:

1. When n is at the end of p, e.g., here $n = 9$:

$$124 \blacksquare 356789$$

2. When the position of the entry n is still a descent after its omission , e.g.,

$$1249 \blacksquare 35678.$$

Therefore there are k choices for the location to insert n into p' (the endpoint plus the $k-1$ descents of p). Hence we conclude that the contribution from this possibility is $kA(n-1, k)$.

- When the omission of n leaves the number of descents decreased by 1. In this case, the resulting permutation is a permutation of $[n-1]$ with $k-2$ many descents, so there are $A(n-1, k-1)$ many of them. Now note that, to reconstruct the permutation p from p', there are two different scenarios:

1. When n is at the front of p, e.g.,

$$9 \blacksquare 124 \blacksquare 35678$$

2. When the position of the entry n is not a descent anymore after its omission , e.g.,

$$12349 \blacksquare 5678.$$

Therefore there are $n-k+1$ choices for the location to insert n into p' (the frontpoint plus the $n-k$ ascents of p'). Hence we conclude that the contribution from this possibility is $(n-k+1)A(n-1, k-1)$.

We have shown that the sides of the theorem count the element of the same set, as desired. $\square$

Remark 16. Read other examples in Section 4.1 not covered in the notes! $\triangle$