Math 184

Lecture Notes Section 3.4 *

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NOTE: The notes is a summary for materials discussed in the class and is not supposed to substitute the textbook. In particular, the proofs here might omit some details for brevity, and are not supposed to be how you write proofs in the exam. Please refer back to the textbook when studying for exams; materials that appear in the textbook but do not appear in the lecture notes might still be tested. Please send me an email if you find typos.

1 Composition of ordinary generating functions

Example 1. Your friendly instructor was once a graduate student, and was forced to write a dissertation. Here are the (fictional) rules[1] that he needed to follows:

1. The dissertation must consist of exactly n pages.
2. The dissertation can have as many chapters as your friendly instructor likes.
3. Each chapter must contain at least one page of text and at least one page of illustrations.

Find the number b_n of ways of arranging the pages of your friendly instructor's dissertation. △

Proof. We start with discussing how many ways *one* chapter with k pages can be arranged. Let a_k be the number of ways one chapter can be arranged. We then have

$$a_0 = 0; \qquad a_k = 2^k - 2 \quad (k \geq 1).$$

Indeed, $a_0 = 0$ and $a_1 = 0$ since a chapter needs to contain at least two pages (one for text, one for illustration). For $k \geq 2$, we have $a_k = 2^k - 2$ because each page can be a text page or an illustration page (hence 2^k), and all-text-chapter or all-illustration-chapter are forbidden (hence -2).

Let $A(x) := \sum_{k=0}^{\infty} a_k x^k$ be the ordinary generating function for a_k's. We want to express $A(x)$ as a "simple"

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[1] The actual rule was actually much more stringent.

function of x, which we achieve by

$$\begin{aligned}
A(x) = &\sum_{k=0}^{\infty} a_k x^k = \sum_{k=1}^{\infty} (2^k - 2) x^k \\
= &\sum_{k=1}^{\infty} 2^k x^k - 2 \sum_{k=1}^{\infty} x^k \\
= &2x \sum_{k=1}^{\infty} 2^{k-1} x^{k-1} - 2x \sum_{k=1}^{\infty} x^{k-1} \\
= &2x \sum_{j=0}^{\infty} 2^j x^j - 2x \sum_{j=0}^{\infty} x^j \qquad \text{(Substitute } j = k-1\text{)} \\
= &\frac{2x}{1-2x} - \frac{2x}{1-x} \\
= &\frac{2x^2}{(1-2x)(1-x)}.
\end{aligned}$$

We now want to find a nice expression of the ordinary generating function $B(x) := \sum_{n=0}^{\infty} b_n x^n$ by using the expression of $A(x)$. Now, let m be the number of chapters of the dissertation.

- Suppose that $m = 0$. Then that means that the instructor had no dissertation to submit and would have been kicked out of the university. Therefore, there is exactly one way to do this (the bad ending) and therefore

  $$B(x) = 1.$$

- Suppose that the dissertation must contain exactly 1 chapter (so $m = 1$). In that case,

  $$b_n = a_n \qquad n \geq 0,$$

  and hence we have

  $$B(x) = A(x).$$

- Suppose that the dissertation must contain exactly 2 chapters (so $m = 2$). In that case,

  $$b_n = \sum_{k=0}^{\infty} a_k a_{n-k} = a_n a_0 + a_{n-1} a_1 + \ldots + a_0 a_n.$$

  Indeed, here k is the number of pages in the first chapter, and $n - k$ is the number of pages in the second chapter. The number k can range from 0 to n (note that in the cases when $k = 0$ and $k = n$ we have $a_k a_{n-k} = 0$). The term a_k counts the number of ways to arrange the first chapter, while the term a_{n-k} counts the number of ways to arrange the second chapter. The formula for b_n now follows from the multiplication and addition principle.

  Now note that, the formula for b_n above is exactly the formula for the coefficient of $A(x)A(x)$! Hence we conclude that

  $$B(x) = A(x)A(x) = (A(x))^2.$$

- Suppose that the dissertation must contain exactly 3 chapters (so $m = 3$). In that case,

$$b_n = \sum_{\substack{k_1,k_2,k_3 \\ k_1+k_2+k_3=n}}^{\infty} a_{k_1}a_{k_2}a_{k_3} = a_n a_0 a_0 + (a_{n-1}a_2a_0 + a_{n-1}a_1a_1 + a_{n-1}a_0a_2) + \ldots + (a_0a_na_0 + a_0a_{n-1}a_1 + \ldots + a_0a_0a_n).$$

Indeed, here k_1 is the number of pages in the first chapter, k_2 is the number of pages in the second chapter, and k_3 is the number of pages in the third chapter. The number k_1, k_2, k_3 can range from 0 to n, and they must sum up to n (since the dissertation has exactly n pages). The term a_{k_1} counts the number of ways to arrange the first chapter, The term a_{k_2} counts the number of ways to arrange the second chapter, and the term a_{k_3} counts the number of ways to arrange the third chapter. The formula for b_n now follows from the multiplication and addition principle.

Now note that, the formula for b_n above is exactly the formula for the coefficient of $A(x)^3$! Indeed, we have

$$\begin{aligned}
A(x)^3 &= A(x)A(x)A(x) \\
&= \left(\sum_{k_1=0}^{\infty} a_{k_1}x^{k_1}\right)\left(\sum_{k_2=0}^{\infty} a_{k_2}x^{k_2}\right)\left(\sum_{k_3=0}^{\infty} a_{k_3}x^{k_3}\right) \\
&= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty} a_{k_1}a_{k_2}a_{k_3}x^{k_1+k_2+k_3} \\
&= \sum_{n=0}^{\infty}\left(\sum_{\substack{k_1,k_2,k_3 \\ k_1+k_2+k_3=n}}^{\infty} a_{k_1}a_{k_2}a_{k_3}\right)x^{k_1+k_2+k_3} \qquad \text{(separating the sum based on the value of } k_1+k_2+k_3\text{)} \\
&= \sum_{n=0}^{\infty}\left(\sum_{\substack{k_1,k_2,k_3 \\ k_1+k_2+k_3=n}}^{\infty} a_{k_1}a_{k_2}a_{k_3}\right)x^{n} \\
&= \sum_{n=0}^{\infty} b_n x^n = B(x).
\end{aligned}$$

Hence we conclude that, when $m = 3$,

$$B(x) = (A(x))^3.$$

- Suppose that the dissertation must contain exactly m chapters. In that case,

$$b_n = \sum_{\substack{k_1,k_2\ldots,k_m \\ k_1+k_2+\ldots+k_m=n}}^{\infty} a_{k_1}a_{k_2}\ldots a_{k_m}.$$

Indeed, here k_i is the number of pages in the i-th chapter (for $i \in \{1,\ldots,m\}$). Each k_i can range from 0 to n, and they must sum up to n (since the dissertation has exactly n pages). The term a_{k_i} counts the number of ways to arrange the i-th chapter. The formula for b_n now follows from the multiplication and addition principle.

Now note that, the formula for b_n above is exactly the formula for the coefficient of $A(x)^m$! Indeed, we have

$$\begin{aligned}
(A(x))^m &= \left(\sum_{k_1=0}^{\infty} a_{k_1} x^{k_1}\right)\left(\sum_{k_2=0}^{\infty} a_{k_2} x^{k_2}\right)\dots\left(\sum_{k_m=0}^{\infty} a_{k_m} x^{k_m}\right) \\
&= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\dots\sum_{k_m=0}^{\infty} a_{k_1} a_{k_2}\dots a_{k_m} x^{k_1+k_2+\ldots+k_m} \\
&= \sum_{n=0}^{\infty}\left(\sum_{\substack{k_1,k_2\ldots,k_m \\ k_1+k_2+\ldots+k_m=n}}^{\infty} a_{k_1} a_{k_2}\dots a_{k_m}\right) x^{k_1+k_2+\ldots+k_m} \qquad \text{(separating the sum based on the value of } k_1+\ldots+ \\
&= \sum_{n=0}^{\infty} b_n x^n = B(x).
\end{aligned}$$

Hence we conclude that, for any fixed m,

$$B(x) = (A(x))^m.$$

Finally, since m can be any arbitrary value, we have from the addition principle that

$$B(x) = \sum_{m=0}^{\infty}(A(x))^m = 1 + A(x) + (A(x))^2 + (A(x))^3 + \dots.$$

Now note that the function in the equation for $B(x)$ above is very familiar! Indeed, it is equal to

$$B(x) = \sum_{m=0}^{\infty}(A(x))^m \frac{1}{1-(A(x))}.$$

We now substitute $A(x) = \frac{2x^2}{(1-2x)(1-x)}$ (which we computed before) to the equation above to get

$$
\begin{aligned}
B(x) = \frac{1}{1-(A(x))} =& \frac{1}{1 - \frac{2x^2}{(1-2x)(1-x)}} \\
=& \frac{1}{1 - \frac{2x^2}{(1-2x)(1-x)}} \\
=& \frac{(1-2x)(1-x)}{(1-2x)(1-x) - 2x^2} \\
=& \frac{(1-2x)(1-x)}{(1-2x-x+2x^2) - 2x^2} \\
=& \frac{(1-2x)(1-x)}{1-3x} \\
=& \frac{2/9}{1-3x} - \frac{6x-7}{9} \qquad \text{(by the magic of partial fraction method)} \\
=& \frac{2}{9}\sum_{n=0}^{\infty} 3^n x^n - \frac{2}{3}x + \frac{7}{9} \\
=& \frac{2}{9}\left(1 + 3x + \sum_{n=2}^{\infty} 3^n x^n\right) - \frac{2}{3}x + \frac{7}{9} \\
=& \left(\frac{2}{9} + \frac{7}{9}\right) + \left(\frac{2}{9}3x - \frac{2}{3}x\right) + \frac{2}{9}\sum_{n=2}^{\infty} 3^n x^n \\
=& 1 + \sum_{n=2}^{\infty} 2 \cdot 3^{n-2} x^n.
\end{aligned}
$$

Hence, we have

$$B(x) = \frac{1}{1-A(x)} - 1 = \sum_{n=2}^{\infty} 2 \cdot 3^{n-2} x^n,$$

which gives us

$$b_n = 2 \cdot 3^{n-2} \quad \text{for } n \geq 2,$$

as desired. □

Remark 2. The novelty of the above computation was that we substituted $y = A(x)$ into the power series $\frac{1}{1-y}$. This is known as the composition of two generating functions. △

Definition 3 (Composition of formal power series)**.** Let $F(x) := \sum_{n=0}^{\infty} f_n x^n$ be a formal power series, and let $A(x) := \sum_{n=0}^{\infty} a_n x^n$ be a formal power series with $a_0 = 0$. Then the *composition* of $F(x)$ and $A(x)$ is the formal power series

$$F(A(x)) := \sum_{n=0}^{\infty} f_n (A(x))^n = f_0 + f_1 A(x) + f_2 (A(x))^2 + \dots. \qquad \triangle$$

Example 4. Let $F(x) = \frac{1}{1-x}$ and $A(x) = 2x - x^2$, then their composition is

$$\begin{aligned} F(A(x)) =& f_0 + f_1 A(x) + f_2(A(x))^2 + \dots \\ =& 1 + A(x) + (A(x))^2 + (A(x))^3 + \dots \\ =& 1 + (2x - x^2) + (2x - x^2)^2 + (2x - x^2)^3 + \dots \\ =& 1 + (2x - x^2) + (4x^2 - 4x^3 + x^4) + (8x^3 - 12x^4 + 6x^5 + x^6) + \dots + \\ =& 1 + 2x + 3x^2 + 4x^3 + \dots . \end{aligned}$$

Computing the exact coefficients directly for the larger term in the formal power series $F(A(x))$ is not easy, as demonstrated from the calculation above. However, the first few coefficients seem familiar, and one might guess that $F(A(x)) = \sum_{n=0}^{\infty}(n+1)x^n$. Indeed, this can actually be shown by the following computation:

$$F(A(x)) = \frac{1}{1-(A(x))} = \frac{1}{1-(2x-x^2)} = \frac{1}{1-2x+x^2} = \frac{1}{(1-x)^2} = \sum_{n=0}^{\infty}(n+1)x^n,$$

as desired. △

Remark 5. You might wonder why we need $a_0 = 0$ in Definition 3; this is best illustrated by the following example. Let $F(x) = \frac{1}{1-x}$ and $A(x) = 1$. Then, by definition,

$$F(A(x)) = \sum_{n=0}^{\infty} f_n(A(x))^n = \sum_{n=0}^{\infty} 1(1)^n = \infty,$$

so the composition is not well-defined! △

The following theorem is the more general version of the ideas presented in Example 1.

Theorem 6. *Suppose that*

- a_k $(k \geq 0)$ *is the number of ways to carry out a certain task on a k-element set, with $a_0 = 0$;*
- b_n *is the number of ways to split the interval $[n]$ into a number of disjoint nonempty* ***ordered*** *subintervals (i.e., $\{1, 2, \dots, c_1\}, \{c_1 + 1, c_1 + 2, \dots, c_2\}, \dots$), then carry out the task above on each subinterval. We adopt the convention that $b_0 = 1$.*

Let $A(x) := \sum_{n=0}^{\infty} a_n x^n$ and $B(x) := \sum_{n=0}^{\infty} b_n x^n$ be the ordinary generating function for a_n and b_n respectively. Then

$$B(x) = \frac{1}{1 - A(x)}.$$

Proof. We left it as an exercise to the reader to prove this theorem. It can be done by using the same idea used in the proof of Example 1. □

2 Composition of exponential generating functions

Example 7. The friendly instructor's cousin is planning a wedding feast with n guests, and your friendly instructor is tasked to partition the n guests ($n \geq 1$) and then have each block of guests sit around a round table. Show that the number of ways this can be done is equal to $n!$, e.g., when $n = 3$, we have $3! = 6$ ways:

$$(123); \qquad (132); \qquad (12)(3); \qquad (13)(2); \qquad (23)(1); \qquad (1)(2)(3).$$

$\triangle$

Proof. Let m be the number of tables used in the wedding party. Suppose that m is fixed. Recall from the exercise we did in Section 3.3 that for the number f_n of partitioning n guests into m **ordered** round tables, the corresponding exponential generating function is

$$F(x) = \left[\ln\left(\frac{1}{1-x}\right)\right]^m.$$

Let h_n be the number of partitioning n guests into m **unordered** round tables (which is what we want to compute!). It then follows from the division principle that

$$h_n = \frac{f_n}{m!}.$$

Hence we have, for fixed value of m,

$$H(x) = \sum_{n=0}^{\infty} h_n \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{f_n}{m!}\frac{x^n}{n!} = \frac{1}{m!}\sum_{n=0}^{\infty} f_n \frac{x^n}{n!} = \frac{1}{m!}F(x) = \frac{1}{m!}\left[\ln\left(\frac{1}{1-x}\right)\right]^m.$$

Note that when $m = 0$, we have $H(x) = 1$. This is because when $m = 0$, there are no tables and the wedding is cancelled; There is exactly one way this can happen, so $H(x) = 1$.

We now let the number of tables m be arbitrary instead of fixed. Then we have from the addition principle that

$$H(x) = \sum_{m=0}^{\infty} \frac{1}{m!}\left[\ln\left(\frac{1}{1-x}\right)\right]^m = 1 + \ln\left(\frac{1}{1-x}\right) + \frac{1}{2!}\left[\ln\left(\frac{1}{1-x}\right)\right]^2 + \frac{1}{3!}\left[\ln\left(\frac{1}{1-x}\right)\right]^3 + \ldots.$$

Let $y := \ln\left(\frac{1}{1-x}\right)$. We then have

$$H(x) = \sum_{m=1}^{\infty} \frac{y^m}{m!} = \sum_{m=0}^{\infty} \frac{y^m}{m!} - 1 = e^y - 1.$$

This then gives us

$$H(x) = e^y - 1 = e^{\ln\left(\frac{1}{1-x}\right)} - 1 = \frac{1}{1-x} - 1 = \sum_{n=1}^{\infty} x^n = \sum_{n=1}^{\infty} n!\frac{x^n}{n!}.$$

By extracting the coefficients of $H(x)$, we then conclude that $h_n = n!$, as desired. $\square$

The following theorem is a more general version of the ideas presented in Example 7.

Theorem 8. *Suppose that*

- a_k $(k \geq 0)$ *is the number of ways to carry out a certain task on a k-element set, with* $a_0 = 0$*;*
- h_n *is the number of ways to partition* $[n]$ *into arbitrary number of* ***unordered*** *blocks, then carry out the task above on each subinterval. We adopt the convention that* $h_0 = 1$.

Let $A(x) := \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$ *and* $H(x) := \sum_{n=0}^{\infty} h_n \frac{x^n}{n!}$ *be the exponential generating function for* a_n *and* h_n *respectively. Then*

$$H(x) = e^{A(x)}.$$

Proof. We left it as an exercise to the reader to prove this theorem. It can be done by using the same idea used in the proof of Example 7. $\square$

Remark 9. It is worth clarifying the the difference between Theorem 6 and Theorem 8 once and for all.

- For Theorem 6 (i.e., the scenario when you use ordinary generating function), the set $[n]$ is split into **ordered subintervals**, e.g., the interval [4] is split into:
  - One subintervals:
    $$(1, 2, 3, 4);$$
  - Two subintervals:
    $$(1), (2, 3, 4); \qquad (1, 2), (3, 4); \qquad (1, 2, 3), (4);$$
  - Three subintervals:
    $$(1), (2), (3, 4); \qquad (1), (2, 3), (4); \qquad (1, 2), (3), (4).$$
  - Four subintervals:
    $$(1), (2), (3), (4).$$

  Here note that the subintervals and its elements are ordered in the increasing manner.
- For Theorem 8 (i.e., the scenario when you use exponential generating function), the set $[n]$ is split into **blocks of subsets**, e.g., [4] is split into
  - One block:
    $$\{1, 2, 3, 4\};$$
  - Two blocks, with one block consists of exactly one element:
    $$\{1\}, \{2, 3, 4\}; \qquad \{2\}, \{1, 3, 4\}; \qquad \{3\}, \{1, 2, 4\}; \qquad \{4\}, \{1, 2, 3\};$$
    $$\{1, 2\}, \{3, 4\}; \qquad \{1, 3\}, \{2, 4\}; \qquad \{1, 4\}, \{2, 3\}.$$

– Three blocks:

$$\{1\},\{2\},\{3,4\}; \qquad \{1\},\{3\},\{2,4\}; \qquad \{1\},\{4\},\{2,3\};$$
$$\{2\},\{3\},\{1,4\}; \qquad \{2\},\{4\},\{1,3\}; \qquad \{3\},\{4\},\{1,2\}.$$

– Four blocks:

$$\{1\},\{2\},\{3\},\{4\}.$$

Here the blocks are unordered, which means that $\{1,3\},\{2,4\}$ is regarded as equal to $\{2,4\},\{1,3\}$. $\triangle$

Remark 10. Read other examples in Section 3.4 not covered in the notes! $\triangle$