Math 184

Lecture Notes Section 3.3 *

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NOTE: The notes is a summary for materials discussed in the class and is not supposed to substitute the textbook. In particular, the proofs here might omit some details for brevity, and are not supposed to be how you write proofs in the exam. Please refer back to the textbook when studying for exams; materials that appear in the textbook but do not appear in the lecture notes might still be tested. Please send me an email if you find typos.

1 Product of ordinary generating functions

Example 1. The friendly instructor is tasked by the higher-up to paint n trees next to the Math Department. For each tree, he randomly chooses one color out of three possibilities: red, blue, and green. At 8pm, he notices that his shift will end soon, and he will not be able to paint more trees that day. In order to celebrate the end of this tyranny, he chooses one of the remaining trees and paints a smiling face on it. How many different looks can the trees of this road have after the overworked instructor has left? $\triangle$

Proof. Let i be the number of trees painted before 8 pm. Note that

- The number i can range from 0 (if the instructor is extremely lazy) to n (if the instructor is extremely efficient);
- These i trees can be painted in 3^i different ways (why?);
- After 8 pm, there are $n-i$ choices of the tree to draw the smiling face.

Therefore, by the combination of addition principle and multiplication principle, the number a_n of ways the trees can be after the instructor left is

$$a_n = \sum_{i=0}^{n} 3^i(n-i) = n + 3(n-1) + 9(n-2) + \dots.$$

(Note that when $i = n$, the term $3^i(n-i)$ is equal to 0. This is because the instructor refuses to paintall the trees as a subtle way to voice his opposition against authority :)) This answers the question, but one might want a more compact, simple answer without a giant summation. We will do so by the generating function.

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As if receiving a divine inspiration, we will instead start with these two formal power series:

$$b_n := 3^n; \qquad B(x) = \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} 3^n x^n = \frac{1}{1-3x};$$

$$d_n := n; \qquad D(x) = \sum_{n=0}^{\infty} d_n x^n = \sum_{n=0}^{\infty} n x^n = x \sum_{n=0}^{\infty} n x^{n-1} = \frac{x}{(1-x)^2}.$$

Now, let's multiply these two power series:

$$\begin{aligned}
B(x)D(x) &= \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} b_i d_{n-i} \right) x^n \\
&= \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} 3^i (n-i) \right) x^n \\
&= \sum_{n=0}^{\infty} a_n x^n \\
&= A(x).
\end{aligned}$$

Hence we have

$$A(x) = \frac{x}{(1-3x)(1-x)^2}.$$

By the magic of the partial fraction method (you NEED to know this before midterm 2!):

$$\begin{aligned}
A(x) &= \frac{3/4}{1-3x} + \frac{-1/4}{1-x} + \frac{-1/2}{(1-x)^2} \\
&= \frac{3}{4} \sum_{n=0}^{\infty} 3^n x^n \; - \frac{1}{4} \sum_{n=0}^{\infty} x^n \; - \frac{1}{2} \sum_{n=0}^{\infty} (n+1) x^n \\
&= \sum_{n=0}^{\infty} \left(\frac{3}{4} 3^n - \frac{1}{4} - \frac{1}{2}(n+1) \right) x^n \\
&= \sum_{n=0}^{\infty} \frac{3^{n+1} - 1 - 2(n+1)}{4} x^n.
\end{aligned}$$

Therefore, the sequence a_n has the following more compact formula:

$$a_n = \frac{3^{n+1} - 1 - 2(n+1)}{4},$$

as desired. $\square$

Remark 2. The key characteristic of the question above is that

> The instructor *started doing something*, then, at one point, *it did something else.*

Problems where the sequence is defined by such a characteristic can be approached by reformulating the generating function as product of two other generating functions. $\triangle$

Theorem 3 (Product formula for ordinary generating functions)**.** *Let f_n, g_n, and h_n be*

- *f_n is the number of ways one can carry out a certain task on the set $[n]$.*

- *g_n is the number of ways one can carry out a different task on the set $[n]$.*

- *h_n is the number of ways one can split $[n]$ into two intervals, carry out the first task on the first interval, and then carry out the second task on the second interval.*

Then

$$H(x) = F(x)G(x),$$

where $F(x), G(x)$, and $H(x)$ are the ordinary generating function for f_n, g_n, and h_n respectively.

Proof. By the definition of h_n, we have

$$h_n = \sum_{i=0}^{n} f_i g_{n-i} = f_0 g_n + f_1 g_{n-1} + \ldots + f_n g_0.$$

It then follows from the multiplication rule for formal power series that

$$H(x) = F(x)G(x),$$

as desired. □

We will know discuss an object in combinatorics with possible the highest rate of appearances in the discrete mathematics literature: the *Catalan numbers*.

Example 4. Two soccer teams, Real Madrid and Barcelona, played an exciting game last weekend. The game ended in a draw, with each team scoring n goals. Throughout the game, it seemed like Real Madrid would win as Barcelona never lead (in terms of number of goals scored). In how many different orders could the $2n$ goals be scored? △

Proof. Let h_n be the number of different orders the $2n$ goals could be scored with Barcelona never leading the game. Let's compute this number for small values of n:

- $n = 0$. In this case the final score was 0-0, and there is exactly one way this scenario could happen, so $h_0 = 1$.

- $n = 1$. In this case the final score was 1-1 and Madrid must score the first goal, so the only scenario is

$$MB,$$

  and $h_1 = 1$.

- $n = 2$. In this case, the possible scenarios are

$$MMBB \qquad MBMB,$$

  so $h_2 = 2$.

- $n = 3$. In this case, the possible scenarios are

$$MMMBBB \qquad MMBMBB \qquad MMBBMB \qquad MBMMBB \qquad MBMBMB,$$

  so $h_3 = 5$.

- $n = 4$. In this case, the possible scenarios are

$$\begin{array}{lllll} MMMMBBBB & MMMBMBBB & MMMBBMBB & MMBMMBBB & MMBMBMBB; \\ MMMBBBMB & MMBMBBMB; & & & \\ MMBBMMBB & MMBBMBMB; & & & \\ MBMMMBBB & MBMMBMBB & MBMMBBMB & MBMBMMBB & MBMBMBMB, \end{array}$$

  so $h_4 = 14$.

From the example above, we notice some patterns:

- Madrid always scored the first goal;
- At some point Barcelona was catching up and evened the score;
- When Barcelona evened the score, it must be Barcelona that scored the goal (duh!).

Let $2i$ be the first time that Barcelona catched up. Here are some examples of the values of i:

$$\begin{array}{ll} MMMMBBBB \ \blacksquare & 2i = 8;\ \text{so } i = 4; \\ MMMBMBBB \ \blacksquare & 2i = 8;\ \text{so } i = 4; \\ MMMBBMBB \ \blacksquare & 2i = 8;\ \text{so } i = 4; \\ MMBMMBBB \ \blacksquare & 2i = 8;\ \text{so } i = 4; \\ MMBMBMBB \ \blacksquare & 2i = 8;\ \text{so } i = 4; \\ MMMBBB \ \blacksquare \ MB & 2i = 6;\ \text{so } i = 3; \\ MMBMBB \ \blacksquare \ MB & 2i = 6;\ \text{so } i = 3; \\ MMBB \ \blacksquare \ MMBB & 2i = 4;\ \text{so } i = 2; \\ MMBB \ \blacksquare \ MBMB & 2i = 4;\ \text{so } i = 2; \\ MB \ \blacksquare \ MMMBBB & 2i = 2;\ \text{so } i = 1; \\ MB \ \blacksquare \ MMBMBB & 2i = 2;\ \text{so } i = 1; \\ MB \ \blacksquare \ MMBBMB & 2i = 2;\ \text{so } i = 1; \\ MB \ \blacksquare \ MBMMBB & 2i = 2;\ \text{so } i = 1; \\ MB \ \blacksquare \ MBMBMB & 2i = 2;\ \text{so } i = 1. \end{array}$$

When Barcelona catched up the first time, Madrid always scored the first goal, and Barcelona always scored the newest goal, so lets cross those two goals out.

$$
\begin{array}{lll}
\cancel{M}MMMBBB\cancel{B} & \blacksquare & i=4;\\
\cancel{M}MMBMBB\cancel{B} & \blacksquare & i=4;\\
\cancel{M}MMBBMB\cancel{B} & \blacksquare & i=4;\\
\cancel{M}MBMMBB\cancel{B} & \blacksquare & i=4;\\
\cancel{M}MBMBMB\cancel{B} & \blacksquare & i=4;\\
\cancel{M}MMBB\cancel{B} & \blacksquare \quad MB & i=3;\\
\cancel{M}MBMB\cancel{B} & \blacksquare \quad MB & i=3;\\
\cancel{M}MB\cancel{B} & \blacksquare \quad MMBB & i=2;\\
\cancel{M}MB\cancel{B} & \blacksquare \quad MBMB & i=2;\\
\cancel{M}\cancel{B} & \blacksquare \quad MMMBBB & i=1;\\
\cancel{M}\cancel{B} & \blacksquare \quad MMBMBB & i=1;\\
\cancel{M}\cancel{B} & \blacksquare \quad MMBBMB & i=1;\\
\cancel{M}\cancel{B} & \blacksquare \quad MBMMBB & i=1;\\
\cancel{M}\cancel{B} & \blacksquare \quad MBMBMB & i=1.
\end{array}
$$

Hence, the strings of goals above can be split into three categories:

- $i = 4$, where we have

$$
\begin{pmatrix}
\cancel{M}MMMBBB\cancel{B}\\
\cancel{M}MMBMBB\cancel{B}\\
\cancel{M}MMBBMB\cancel{B}\\
\cancel{M}MBMMBB\cancel{B}\\
\cancel{M}MBMBMB\cancel{B}
\end{pmatrix}
\qquad \blacksquare \qquad \varnothing.
$$

  This is $h_3 \times h_0$.

- $i = 3$, where we have

$$
\begin{pmatrix}
\cancel{M}MMBB\cancel{B}\\
\cancel{M}MBMB\cancel{B}
\end{pmatrix}
\qquad \blacksquare \qquad
\begin{pmatrix} MB \end{pmatrix}.
$$

  This is $h_2 \times h_1$.

- $i = 2$, so we have

$$
\begin{pmatrix} \cancel{M}MB\cancel{B} \end{pmatrix}
\qquad \blacksquare \qquad
\begin{pmatrix}
MMBB\\
MBMB
\end{pmatrix}
$$

  This is $h_1 \times h_2$.

- $i = 1$, so we have

$$\left(\cancel{M}\cancel{B}\right) \qquad \blacksquare \qquad \begin{pmatrix} MMMBBB \\ MMBMBB \\ MMBBMB \\ MBMMBB \\ MBMBMB \end{pmatrix}.$$

  This is $h_0 \times h_3$.

Hence we conclude that

$$h_4 = h_3 h_0 + h_2 h_1 + h_1 h_2 + h_0 h_3.$$

By the same reasoning, we have, for $n \geq 1$,

$$h_{n+1} = \sum_{i=0}^{n} h_i h_{n-i} = \sum_{i=0}^{n} h_{n-i} h_i = h_n h_0 + h_{n-1} h_1 + \ldots + h_1 h_{n-1} + h_0 h_n.$$

(Read the textbook for the rigorous proof of this recurrence relation!)

Now that we have this recurrence relation, let's plug this into our generating function:

$$\begin{aligned} H(x) =& \sum_{n=0}^{\infty} h_n x^n \\ =& 1 + \sum_{n=1}^{\infty} h_n x^n \\ =& 1 + x \sum_{n=1}^{\infty} h_n x^{n-1} \\ =& 1 + x \sum_{m=0}^{\infty} h_{m+1} x^m \qquad \text{(Substitute } m = n - 1\text{);} \\ =& 1 + x \sum_{m=0}^{\infty} \left(\sum_{i=0}^{m} h_i h_{m-i} \right) x^m \\ =& 1 + xH(x)H(x) \\ =& 1 + xH(x)^2. \end{aligned}$$

So we have the following formula:

$$H(x) = 1 + xH(x)^2,$$

which is the same as

$$0 = xH(x)^2 - H(x) + 1.$$

This is a quadratic equation for $H(x)$. Solving it by using quadratic formula (remember this from middle school!):

$$ay^2 + by + c = 0 \quad \text{has solutions} \quad y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

so by setting $y = H(x)$, we get

$$H(x) = \frac{1 \pm \sqrt{1 - 4x}}{2x}.$$

To determine to take the plus scenario or minus scenario, we use the following argument

- First, $H(0) = 1$.
- Now, suppose that $H(x) = \frac{1+\sqrt{1-4x}}{2x}$. Then
$$H(0) = \frac{1+1}{2(0)} = \infty,$$
so we get a contradiction!
- Now, suppose that $H(x) = \frac{1-\sqrt{1-4x}}{2x}$. Then
$$H(0) = \frac{1-1}{2(0)} = \frac{0}{0} = \lim_{x\to 0} \frac{\frac{\partial}{dx} 1 - \sqrt{1-4x}}{\frac{\partial}{dx} 2x} = \lim_{x\to 0} \frac{2(1-4x)^{-1/2}}{2} = 1.$$
So things are fine!

Hence we conclude that
$$H(x) = \frac{1-\sqrt{1-4x}}{2x}.$$
Now, let's take the Taylor expansion of $\sqrt{1-4x}$,
$$\sqrt{1-4x} = 1 + \sum_{n=1}^{\infty} \binom{1/2}{n} (-4x)^n = 1 + \sum_{n=1}^{\infty} \binom{1/2}{n} (-4)^n x^n.$$
Now note that, by definition, for $n \geq 1$,
$$\begin{aligned}\binom{1/2}{n} = \frac{\frac{1}{2} \cdot \frac{-1}{2} \cdot \frac{-3}{2} \cdots \frac{-2n+3}{2}}{n!} &= (-1)^{n-1} \frac{1 \cdot 1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n-3)}{2^n \cdot n!} \\ &= (-1)^{n-1} \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdots (2n-2)}{2^n \cdot n!} \cdot \frac{1}{2 \cdot 4 \cdot 6 \cdots (2n-2)} \\ &= (-1)^{n-1} \frac{(2n-2)!}{2^n \cdot n!} \frac{1}{2^{n-1}(n-1)!} \\ &= (-1)^{n-1} \frac{(2n-2)!}{2^n \cdot (n-1)!\, n} \frac{1}{2^{n-1}(n-1)!} \\ &= (-1)^{n-1} \binom{2n-2}{n-1} \frac{1}{n 2^{2n-1}}.\end{aligned}$$
Plugging this into the equation above, we get
$$\begin{aligned}\sqrt{1-4x} = 1 + \sum_{n=1}^{\infty} \binom{1/2}{n} (-4)^n x^n &= 1 + \sum_{n=1}^{\infty} \left((-1)^{n-1} \binom{2n-2}{n-1} \frac{1}{n 2^{2n-1}} \right) (-4)^n x^n \\ &= 1 - 2 \sum_{n=1}^{\infty} \frac{\binom{2n-2}{n-1}}{n} x^n.\end{aligned}$$
Hence we have
$$\begin{aligned}H(x) &= \frac{1-\sqrt{1-4x}}{2x} = \frac{1 - \left(1 - 2\sum_{n=1}^{\infty} \frac{\binom{2n-2}{n-1}}{n} x^n\right)}{2x} = \sum_{n=1}^{\infty} \frac{\binom{2n-2}{n-1}}{n} x^{n-1} = \sum_{m=0}^{\infty} \frac{\binom{2m}{m}}{m+1} x^m \\ &= \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{n+1} x^n.\end{aligned}$$

Hence we conclude that

$$h_n = \frac{\binom{2n}{n}}{n+1}.$$

This is the formula for the Catalan number we want to find. $\square$

2 Product of exponential generating functions

Example 5. Let d_n be the sequence defined by

$$d_n := \sum_{i=0}^{n} \binom{n}{i} \cdot (3^i i!) \cdot (4^{n-i}(n-i)!).$$

Find a closed-form formula for d_n. $\triangle$

Proof. Let $D(x) := \sum_{n=0}^{\infty} d_n \frac{x^n}{n!}$ be the exponential generating function for d_n's. By another divine inspiration, we define sequences a_n and b_n

$$a_n := 3^n n!; \qquad A(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} = \sum_{n=0}^{\infty} 3^n n! \frac{x^n}{n!} = \sum_{n=0}^{\infty} 3^n x^n = \frac{1}{1-3x};$$
$$b_n := 4^n n!; \qquad B(x) = \sum_{n=0}^{\infty} b_n \frac{x^n}{n!} = \sum_{n=0}^{\infty} 4^n n! \frac{x^n}{n!} = \sum_{n=0}^{\infty} 4^n x^n = \frac{1}{1-4x}.$$

Let's take the product of $A(x)$ and $B(x)$:

$$\begin{aligned}
A(x)B(x) &= \left(\sum_{n=0}^{\infty} \frac{a_n}{n!} x^n\right)\left(\sum_{n=0}^{\infty} \frac{b_n}{n!} x^n\right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} \frac{a_i}{i!} \frac{b_{n-i}}{(n-i)!}\right) x^n \\
&= \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} \frac{a_i}{i!} \frac{b_{n-i}}{(n-i)!} \frac{n!}{n!}\right) x^n \\
&= \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} \binom{n}{i} a_i b_{n-i}\right) \frac{x^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} \binom{n}{i} (3^i i!)(4^{n-i}(n-i)!)\right) \frac{x^n}{n!} \\
&= \sum_{n=0}^{\infty} d_n \frac{x^n}{n!} \\
&= D(x).
\end{aligned}$$

Therefore, we get an explicit formula for $D(x)$, namely,

$$D(x) = A(x)B(x) = \frac{1}{1-3x} \cdot \frac{1}{1-4x}.$$

Again by using the magic of partial fractions method, we get

$$\begin{aligned}D(x) = \frac{4}{1-4x} - \frac{3}{1-3x} =& 4\sum_{n=0}^{\infty} 4^n x^n - 3\sum_{n=0}^{\infty} 3^n x^n \\ =& \sum_{n=0}^{\infty}(4^{n+1} - 3^{n+1})x^n \\ =& \sum_{n=0}^{\infty} n!(4^{n+1} - 3^{n+1})\frac{x^n}{n!}.\end{aligned}$$

Therefore, we conclude that

$$d_n = n!(4^{n+1} - 3^{n+1}),$$

as desired. □

Theorem 6 (Product formula for exponential generating functions)**.** *Let f_n, g_n, and h_n be*

- *f_n is the number of ways one can carry out a certain task on the set $[n]$.*
- *g_n is the number of ways one can carry out a different task on the set $[n]$.*
- *h_n is the number of ways to choose a subset S of $[n]$, carry out the first task on S, and then carry out the second task on the set $[n] - S$.*

Then

$$H(x) = F(x)G(x),$$

where $F(x), G(x)$, and $H(x)$ are the exponential generating function for f_n, g_n, and h_n respectively.

Proof. By the definition of h_n, we have

$$h_n = \sum_{i=0}^{n}\binom{n}{i} f_i g_{n-i}.$$

On the other hand, note that

$$\begin{aligned}F(x)G(x) =& \left(\sum_{n=0}^{\infty}\frac{f_n}{n!}x^n\right)\left(\sum_{n=0}^{\infty}\frac{g_n}{n!}x^n\right) \\ =& \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\frac{f_i}{i!}\frac{g_{n-i}}{(n-i)!}\right)x^n \\ =& \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\frac{f_i}{i!}\frac{g_{n-i}}{(n-i)!}\frac{n!}{n!}\right)x^n \\ =& \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\binom{n}{i}f_i g_{n-i}\right)\frac{x^n}{n!} \\ =& \sum_{n=0}^{\infty} h_n\frac{x^n}{n!} \\ =& H(x).\end{aligned}$$

as desired. □

Example 7. The friendly instructor's cousin is planning a wedding feast with n guests, and your friendly instructor is tasked to partition the n guests into two round tables, one for treasured friends and one for mortal enemies (both tables cannot be empty!). Let h_n be the number of options that the friendly instructor has. Compute the exponential generating function $H(x) := \sum_{n=0}^{\infty} h_n \frac{x^n}{n!}$ for h_n's. $\triangle$

Proof. Let f_k be the number of ways to arrange k guests (treasured friends) into a round table. We have learned from Section 1 that

$$f_0 = 0; \qquad f_k = (k-1)! \qquad \text{for } k \geq 1.$$

Note that here $f_0 = 0$ because we assume that the cousin has at least one treasured friend. Let $F(x) := \sum_{k=0}^{\infty} f_k \frac{x^k}{k!}$ be the exponential generating function of f_k's. We have

$$F(x) = \sum_{k=0}^{\infty} f_k \frac{x^k}{k!} = \sum_{k=1}^{\infty} f_k \frac{x^k}{k!} = \sum_{k=1}^{\infty} (k-1)! \frac{x^k}{k!} = \sum_{k=1}^{\infty} \frac{x^k}{k} = \ln\left(\frac{1}{1-x}\right).$$

Let g_k be the number of ways to arrange k guests (mortal enemies) into a round table. Let $G(x) := \sum_{k=0}^{\infty} g_k \frac{x^k}{k!}$ be the exponential generating function of g_k's. By the same argument as for $F(x)$, we conclude that

$$G(x) = \ln\left(\frac{1}{1-x}\right).$$

Now note that, when assigning the seats for n guests, the friendly instructor needs to assign some of the guests into the first table, the rest into the second table, and then put each group into a round table. This is exactly the scenario in Theorem 6. Hence it follows that

$$H(x) = F(x)G(x) = \left[\ln\left(\frac{1}{1-x}\right)\right]^2,$$

as desired. $\square$

Exercise 8. Suppose that the friendly instructor needs to fill m tables instead of 2 tables: the first table consisting of closest friends of the first degree, the second table consisting of closest friends of the second degree, and so on (That is to say, the tables are ordered). Assume that every table must be nonempty[1]. By modifying the argument in Theorem 6, show that the corresponding exponential generating function is

$$H(x) = \left[\ln\left(\frac{1}{1-x}\right)\right]^m. \qquad \triangle$$

Remark 9. Read other examples in Section 3.3 not covered in the notes! $\triangle$

[1]After all, each wedding table costs a lot of money (from real life experience), so it makes no sense to have an empty table.