Math 184

Lecture Notes Section 3.2 *

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NOTE: The notes is a summary for materials discussed in the class and is not supposed to substitute the textbook. In particular, the proofs here might omit some details for brevity, and are not supposed to be how you write proofs in the exam. Please refer back to the textbook when studying for exams; materials that appear in the textbook but do not appear in the lecture notes might still be tested. Please send me an email if you find typos.

1 Ordinary generating function

Definition 1 (Ordinary generating function)**.** Given a sequence $(a_0, a_1, a_2, \dots,)$ of real numbers, its *ordinary generating function* is the formal power series

$$\sum_{n=0}^{\infty} a_n x^n. \qquad \triangle$$

Example 2. Consider the sequence $(a_i)_{n\geq 0}$ given by

$$a_n = 1 \qquad \text{for all } n \geq 0.$$

The corresponding ordinary generating function is

$$A(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}. \qquad \triangle$$

Example 3. Consider the sequence $(a_n)_{n\geq 0}$ given by

$$a_n = n+1 \qquad \text{for all } n \geq 0.$$

The corresponding ordinary generating function is

$$A(x) = \sum_{n=0}^{\infty} (n+1)x^n = \frac{1}{(1-x)^2}. \qquad \triangle$$

Remark 4. The following is an apt description of generating function by Herbert Wilf:

"A generating function is a clothesline on which we hang up a sequence of numbers for display."

$\triangle$

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2 Extraction of coefficients

Definition 5. Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$ be a formal power series. We write

$$[x^n]A(x) \;:=\; a_n,$$

the coefficient of x^n in the formal power series $A(x)$. △

Example 6. Let $A(x)$ be the formal power series $A(x) = 1 - x$. Then

$$[x^n]A(x) = \begin{cases} 1, & \text{if } n = 0; \\ -1, & \text{if } n = 1; \\ 0, & \text{if } n \geq 2. \end{cases}$$ △

Example 7. Let $A(x)$ be the formal power series $A(x) = \frac{1}{1-x}$ (the multiplicative inverse of $1 - x$). Since we have known that

$$A(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n,$$

we then have

$$[x^n]A(x) = 1 \qquad \text{for } n \geq 0.$$ △

One of the initial reason people started studying generating function is because it can be used to solve recurrence relations.

Example 8. Consider the sequence $(a_n)_{n \geq 0}$ given by

$$a_0 := 5; \qquad a_n := 3a_{n-1} - 1 \qquad \text{for } n \geq 1.$$

Give an explicit formula for a_n. △

Proof. Let $A(x)$ be the ordinary generating function of this series. Then

$$\begin{aligned}
A(x) &= \sum_{n=0} a_n x^n \;=\; 5 + \sum_{n=1}^{\infty} a_n x^n \\
A(x) &= 5 + \sum_{n=1}^{\infty} (3a_{n-1} - 1)x^n \\
A(x) &= 5 + x\sum_{n=1}^{\infty} (3a_{n-1} - 1)x^{n-1} \\
A(x) &= 5 + x\sum_{j=0}^{\infty} (3a_j - 1)x^j \qquad \text{(Substitute } j = n-1\text{)} \\
A(x) &= 5 + 3x\sum_{j=0}^{\infty} a_j x^j - x\sum_{j=0} x^j \\
A(x) &= 5 + 3xA(x) - \frac{x}{1-x} \\
A(x)(1-3x) &= 5 - \frac{x}{1-x} \\
A(x) &= \frac{5}{1-3x} - \frac{x}{(1-x)(1-3x)}.
\end{aligned}$$

We now expand the series of each term in the right side separately. First we have

$$\frac{5}{1-3x} \;=\; 5\sum_{n=0}^{\infty} 3^n x^n \;=\; \sum_{n=0}^{\infty}(5\cdot 3^n)x^n.$$

Next we have

$$\begin{aligned}
\frac{x}{(1-x)(1-3x)} &= \frac{1}{2}\frac{1}{1-3x} - \frac{1}{2}\frac{1}{1-x} \qquad \text{(by the method of partial fraction)}\\
&= \frac{1}{2}\sum_{n=0}^{\infty} 3^n x^n - \frac{1}{2}\sum_{n=0}^{\infty} x^n\\
&= \sum_{n=0}^{\infty}\frac{3^n-1}{2}x^n.
\end{aligned}$$

Hence we have

$$A(x) = \sum_{n=0}^{\infty}\left(5\cdot 3^n - \frac{3^n-1}{2}\right)x^n,$$

and the explicit formula for a_n is then given by

$$a_n = [x^n]A(x) = 5\cdot 3^n - \frac{(3^n-1)}{2}.$$

This completes the proof. $\square$

Remark 9. We can also use induction to prove that the sequence $(a_n)_{n\geq 00}$ in Example 7 satisfies the formula

$$a_n = 5\cdot 3^n - \frac{(3^n-1)}{2},$$

assuming we manage to guess the formula in the first place. However, one advantage of the generating function method is that we do not even need to guess the right formula! The answer just comes out at the end of the calculation, which is one advantage that the induction method does not have. $\triangle$

3 Exponential generating function

Example 10. Consider the sequence $(a_n)_{n\geq 0}$ given by

$$a_0 := 5; \qquad a_n := na_{n-1} + 2n \qquad \text{for } n \geq 1.$$

Give an explicit formula for a_n. $\triangle$

We can try to solve this question with the ordinary generating function method. Let $A(x)$ be the corre-

sponding generating function. Then

$$\begin{aligned} A(x) &= \sum_{n=0}^{\infty} a_n x^n \\ &= 5 + \sum_{n=1}^{\infty} (n a_{n-1} x^n + 2n x^n) \\ &= 5 + x \sum_{j=0}^{\infty} (j+1) a_j x^j + 2x \sum_{j=0}^{\infty} x^j \\ &= 5 + x \sum_{j=0}^{\infty} (j+1) a_j x^j + \frac{2x}{(1-x)^2}. \end{aligned}$$

You can try to do some manipulation to the equation above, but then you would notice that you would just go around circles and could never get rid of the term $\sum_{j=0}^{\infty}(j+1)a_j x^j$ (or its variation)!
We therefore requires a new approach.

Definition 11 (Exponential generating function)**.** Given a sequence $(a_0, a_1, a_2, \ldots,)$ of real numbers, its *exponential generating function* is the formal power series

$$\sum_{n=0}^{\infty} a_n \frac{x^n}{n!}.$$

△

Example 12. Consider the sequence $(a_n)_{n \geq 0}$ given by

$$a_n := 1 \qquad \text{for } n \geq 0.$$

Then the exponential generating function $A(x)$ of this series is

$$A(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x.$$

△

We will now use exponential generating function to give an answer to Example 10.

Answer to Example 10. Let $D(x)$ be the exponential generating function corresponding to the sequence $(a_n)_{n \geq 0}$ in Example 10. We have

$$\begin{aligned} D(x) &= \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} \\ &= 5 + \sum_{n=1}^{\infty} n a_{n-1} \frac{x^n}{n!} + 2 \sum_{n=1}^{\infty} n \frac{x^n}{n!} \\ &= 5 + \sum_{n=1}^{\infty} a_{n-1} \frac{x^n}{(n-1)!} + 2 \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!} \\ &= 5 + x \sum_{n=1}^{\infty} a_{n-1} \frac{x^{n-1}}{(n-1)!} \ + \ 2x \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} \\ &= 5 + x \sum_{m=0}^{\infty} a_m \frac{x^m}{m!} \ + \ 2x \sum_{m=0}^{\infty} \frac{x^m}{m!} \qquad \text{(substitute } m = n-1\text{)} \\ &= 5 + xD(x) + 2xe^x. \end{aligned}$$

Therefore,

$$D(x) = \frac{5}{1-x} + \frac{2x}{1-x}e^x.$$

We now write out the formal power series for each term in the right side of the equation. First note that

$$\frac{5}{1-x} = 5\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} 5 \cdot n! \frac{x^n}{n!}.$$

Then note that

$$\begin{aligned}
\frac{2x}{1-x}e^x &= \left(2x\sum_{n=0}^{\infty} x^n\right)\left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right) \\
&= 2x\left(\sum_{n=0}^{\infty} x^n\right)\left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right) \\
&= 2x\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n} \frac{1}{i!}\right)x^n \\
&= 2\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n} \frac{1}{i!}\right)x^{n+1} \\
&= 2\sum_{m=1}^{\infty}\left(\sum_{i=0}^{m-1} \frac{1}{i!}\right)x^m \qquad \text{(Substitute } m = n+1\text{)} \\
&= 2\sum_{m=1}^{\infty}\left(\sum_{i=0}^{m-1} \frac{m!}{i!}\right)\frac{x^m}{m!} \\
&= \sum_{m=1}^{\infty}\left(2\sum_{i=0}^{m-1} \frac{m!}{i!}\right)\frac{x^m}{m!} \\
&= \sum_{m=0}^{\infty}\left(2\sum_{i=0}^{m-1} \frac{m!}{i!}\right)\frac{x^m}{m!}.
\end{aligned}$$

where the last equality is because $\sum_{i=0}^{m-1} f(i)$ is equal to 0 when $m = 0$. (Recall that $\sum_{i=0}^{m-1}$ means that i ranges over all integers i satisfying $0 \le i \le m-1$. When $m = 0$, this means we are looking for integers satisfying $0 \le i \le -1$, which obviously does not exist, and hence the sum is an empty sum, which is equal to 0).

Therefore we have

$$D(x) = \sum_{n=0}^{\infty}\left(5n! + 2\sum_{i=0}^{n-1} \frac{n!}{i!}\right)\frac{x^n}{n!}.$$

Since a_n is the coefficient of $x^n/n!$ in $D(x)$, we conclude that

$$a_n = 5n! + 2\sum_{i=0}^{n-1} \frac{n!}{i!},$$

as desired. □