Math 184

Lecture Notes Section 3.1 *

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NOTE: The notes is a summary for materials discussed in the class and is not supposed to substitute the textbook. In particular, the proofs here might omit some details for brevity, and are not supposed to be how you write proofs in the exam. Please refer back to the textbook when studying for exams; materials that appear in the textbook but do not appear in the lecture notes might still be tested. Please send me an email if you find typos.

1 Generating functions: motivation

Example 1 (Fibonacci, ancient India)**.** A newly born breeding pair of rabbits are put in a field; each breeding pair mates at the age of one month, and at the end of their second month they always produce another pair of rabbits. Rabbits never die, and they continue breeding forever. How many pairs will there be in n months? △

Let F_n be the number of rabbit pairs at the end of the n-th month.

- In the beginning, we just put the original breeding pair of rabbits into the field. This means $F_0 = 1$.
- At the end of the first month, they mate, but there is still only 1 pair. This means $F_1 = 1$.
- At the end of the second month they produce a new pair, so there are two pairs in the field. This means $F_2 = 2$.
- At the end of the third month, the original pair produce a second pair, but the second pair only mate without breeding, so there are 3 pairs in all. This means $F_3 = 3$.
- At the end of the fourth month, the original pair has produced yet another new pair, and the pair born two months ago also produces their first pair, making 5 pairs. This means $F_4 = 5$.
- At the end of the n-th month, the number of pairs of rabbits is equal to the number of mature pairs (that is, the number of pairs in month $n-2$) plus the number of pairs alive last month (month $n-1$) This means $F_n = F_{n-1} + F_{n-2}$.

Written in a concise mathematical form, this is the famous *Fibonacci sequence* given by

$$F_0 = F_1 = 1; \qquad F_n = F_{n-1} + F_{n-2} \quad (n \geq 2).$$

The earliest reference to Fibonacci sequence can be traced back to ancient India in the work of Acharya Pingala (3rd centure BCE). It has been rediscovered several times independently by different mathematicians from different eras and locations. In particular, Fibonacci (12th century) came up with the example given in the beginning.

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This number has the following closed form formula

$$F_n = \frac{1}{\sqrt{5}}\left(-\left(\frac{1-\sqrt{5}}{2}\right)^{n+1} + \left(\frac{1+\sqrt{5}}{2}\right)^{n+1}\right). \tag{1}$$

This formula is a product of the Age of Enlightenment, and was credited to different mathematicians, such as Binet, de Moivre, and Bernoulli. We would give a non-rigorous proof of this formula today.

Remark 2. One remarkable aspect of (1) is that it is not immediate from the formula if F_n is even a rational number! Try to compute the first few Fibonacci numbers using this formula to convince you that this formula indeed always yield a positive integer. $\triangle$

Let $F(x)$ be a function given by

$$F(x) := \sum_{n=0}^{\infty} F_n x^n = 1 + x + 2x^2 + 3x^3 + 5x^4 + \dots.$$

Note that it is not immediate from the definition that $F(x)$ is well-defined (as the sum might not converge), but at this point let us just assume that the function is well-defined for "small" values of x.
We now consider the following function:

$$(1 - x - x^2)F(x),$$

and let's try to compute this function. Note that

$$\begin{aligned}
&(1 - x - x^2)F(x) \\
=&(1 - x - x^2)\sum_{n=0}^{\infty} F_n x^n \\
=&\sum_{n=0}^{\infty} F_n x^n - \sum_{n=0}^{\infty} F_n x^{n+1} - \sum_{n=0}^{\infty} F_n x^{n+2} \\
=&(1 + x + 2x^2 + 3x^3 + \dots + a_n x^n + \dots) - (x + x^2 + 2x^3 + 3x^4 + \dots + a_{n-1}x^n + \dots) - (x^2 + x^3 + 2x^4 + \dots + a_{n-2}x^n + \dots) \\
=&1 + (1-1)x + (2-1-1)x^2 + (3-2-1)x^3 + (5-3-2)x^4 + \dots + (a_n - a_{n-1} - a_{n-2})x^n + \dots \ =
\end{aligned}$$

This implies that

$$F(x) = \frac{1}{1 - x - x^2}.$$

Factoring the roots of the polynomial, we get

$$1 - x - x^2 = (\varphi + x)\left(\varphi^{-1} - x\right),$$

where $\varphi := \frac{1+\sqrt{5}}{2}$ is the famous *Golden ratio*. We then have

$$\begin{aligned}
F(x) =& \frac{1}{(\varphi + x)} \frac{1}{(\varphi^{-1} - x)} \\
=& \frac{1}{\varphi^{-1} + \varphi} \left(\frac{1}{\varphi + x} + \frac{1}{\varphi^{-1} - x} \right) \qquad \text{(by partial fraction method)} \\
=& \frac{1}{\sqrt{5}} \left(\frac{\varphi^{-1}}{1 + (\varphi^{-1} x)} + \frac{\varphi}{1 - (\varphi x)} \right) \\
=& \frac{1}{\sqrt{5}} \left(\varphi^{-1} \sum_{n=0}^{\infty} (-\varphi^{-1})^n x^n + \varphi \sum_{n=0}^{\infty} \varphi^n x^n \right) \\
=& \sum_{n=0}^{\infty} \frac{1}{\sqrt{5}} \left(-(-\varphi^{-1})^{n+1} + \varphi^{n+1} \right) x^n \\
=& \sum_{n=0}^{\infty} \frac{1}{\sqrt{5}} \left(- \left(\frac{1 - \sqrt{5}}{2} \right)^{n+1} + \left(\frac{1 + \sqrt{5}}{2} \right)^{n+1} \right) x^n.
\end{aligned}$$

This formula allows us to conclude that

$$F_n = \frac{1}{\sqrt{5}} \left(- \left(\frac{1 - \sqrt{5}}{2} \right)^{n+1} + \left(\frac{1 + \sqrt{5}}{2} \right)^{n+1} \right),$$

as desired.

2 Formal power series

We now build a formal foundation to the techniques we used before. Recall from Calculus that the *power series* of a function is a way to write a function $f(x)$ as

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots,$$

for some values of x contained in a "nice" set.
Some example of power series:

- $$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots \qquad (|x| < 1);$$
- $$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} n x^{n-1} = 1 + 2x + 3x^2 + \dots \qquad (|x| < 1);$$
- $$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots \qquad x \in \mathbb{R}.$$

Definition 3. A *formal power series* is an infinite sequence

$$(a_0, a_1, a_2, a_3, \dots), \tag{2}$$

but is written in the form

$$A(x) := \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots. \tag{3}$$

$\triangle$

Remark 4. Here the symbol x is just a symbol, it does NOT denote *any value*, not even *any unknown value*. This is done so that we can ignore the question of convergence entirely. The reason why we write the formal power series in the notation of (3) rather than (2) is to give us the intuition for the next two operations. $\triangle$

Definition 5 (Addition of formal power series)**.** Let $A(x) := \sum_{n=0}^{\infty} a_n x^n$ and $B(x) := \sum_{n=0}^{\infty} b_n x^n$ be two formal power series. Their *addition* $(A + B)(x)$ is equal to

$$(A + B)(x) := \sum_{n=0}^{\infty} (a_n + b_n) x^n. \qquad \triangle$$

Note that, in the notation of (2), the addition of two formal power series is the infinite sequence

$$(a_0 + b_0, a_1 + b_1, a_2 + b_2, \ldots).$$

This definition is motivated by what the addition of two infinite series will look like *if* $A(x)$ and $B(x)$ are two power series of some "nice functions".

Definition 6 (Multiplication)**.** Let $A(x) := \sum_{n=0}^{\infty} a_n x^n$ and $B(x) := \sum_{n=0}^{\infty} b_n x^n$ be two formal power series. Their *multiplication* $(A \cdot B)(x)$ is equal to

$$(A \cdot B)(x) := \sum_{n=0}^{\infty} (a_0 b_n + a_1 b_{i-1} + a_2 b_{i-2} + \ldots + a_n b_0) x^n. \qquad \triangle$$

Note that, in the notation of (2), the multiplication of two formal power series is the infinite sequence

$$(a_0 b_0, a_0 b_1 + a_1 b_0, a_0 b_2 + a_1 b_1 + a_2 b_0, \ldots).$$

This definition is motivated by what the product of two infinite power series will look like *if* $A(x)$ and $B(x)$ are two power series of some "nice functions". Indeed, when you take the product of two power series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$, the coefficient of x^n in the product is

$$\begin{array}{ccccccccc} a_0 & +a_1 x & + & a_2 x^2 & + & \ldots & + & a_n x^n \\ b_n x^n & +b_{n-1} x^{n-1} & + & b_{n-2} x^{n-2} & + & \ldots & + & b_0 \end{array},$$

which indeed is equal to

$$a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \ldots + a_n b_0.$$

Exercise 7. Let $A(x)$ be the formal power series $\sum_{n=0}^{\infty} x^n$. Check that the product $B(x) = (A \cdot A)(x)$ is given by

$$B(x) = \sum_{n=0}^{\infty} (n + 1) x^n.$$

In particular, note that, if x is a real number in $(-1, 1)$ instead of a symbol, then $A(x)$ would be the power series of the function $\frac{1}{1-x}$, and in this case the product $B(x) = (A \cdot A)(x)$ would be the power series of the function $\frac{1}{(1-x)^2}$, which is indeed the case as demonstrated above. $\triangle$

One important property of the power series multiplication is that it is a commutative operation.

Lemma 8. *Let* $A(x) := \sum_{n=0}^{\infty} a_n x^n$ *and* $B(x) := \sum_{n=0}^{\infty} b_n x^n$ *be two formal power series. Then*

$$(A \cdot B)(x) \;=\; (B \cdot A)(x)$$

Proof. The proof is left as an exercise. $\square$

Definition 9 (Formal derivative)**.** Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$ be a formal power series. The *formal derivative* $A'(x)$ is

$$A'(x) := \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n. \qquad \triangle$$

Exercise 10. Let $A(x)$ be the formal power series $\sum_{n=0}^{\infty} x^n$. Check that the derivative $A'(x)$ is given by

$$A'(x) = \sum_{n=0}^{\infty} (n+1) x^n = 1 + 2x + 3x^2 + \dots.$$

In particular, note that, if x is a real number in $(-1, 1)$ instead of a symbol, then $A(x)$ would be the power series of the function $\frac{1}{1-x}$, and in this case the product $A'(x)$ would be the power series of the function $\frac{1}{(1-x)^2}$, which is indeed the case as demonstrated above. $\triangle$

Definition 11 (Multiplicative inverse)**.** Let $A(x) := \sum_{n=0}^{\infty} a_n x^n$ be a formal power series. The *multiplicative inverse* $A^{-1}(x)$, **if it exists**, is the formal power series $A^{-1}(x)$ such that

$$(A \cdot A^{-1})(x) = (A^{-1} \cdot A)(x) = 1.$$

That is to say, the infinite sequence that corresponds to $(A^{-1} \cdot A)(x)$ is

$$(1, 0, 0, \dots, 0). \qquad \triangle$$

Example 12. Let $A(x)$ be the formal power series $\sum_{n=0}^{\infty} x^n$. One can check that the multiplicative inverse $A^{-1}(x)$ is equal to $1 - x$. Indeed, let $(a_n)_{n \geq 0}$ and $(b_n)_{n \geq 0}$ be the infinite series that corresponds to $A(x)$ and $A^{-1}(x)$, respectively. Note that

$$\begin{aligned} & & a_n := 1 \qquad & \text{for } n \geq 0; \\ b_0 = 1; \qquad & b_1 = -1; \qquad & b_n = 0 \qquad & \text{for } n \geq 2. \end{aligned}$$

Let $(c_n)_{n \geq 0}$ be the infinite series that corresponds to $(A \cdot A^{-1})(x)$. Then it follows from the multiplication of formal power series that

$$\begin{aligned} c_0 &= a_0 b_0 = (1)(1) = 1; \\ c_1 &= a_0 b_1 + b_0 a_1 = (1)(-1) + (1)(1) = 0; \\ c_n &= a_0 b_n + a_1 b_{n-1} + \dots + a_{n-2} b_2 + a_{n-1} b_1 + a_n b_0 = (1)(0) + (1)(0) + \dots + (1)(0) + (1)(-1) + (1)(1) = 0. \end{aligned}$$

It then follows that $(A \cdot A^{-1})(x) = 1$.

In particular, note that, if x is a real number in $(-1, 1)$ instead of a symbol, then $A(x)$ would be the power series of the function $\frac{1}{1-x}$, and in this case the multiplicative inverse $A^{-1}(x)$ would be the power series of the function $1 - x$, which is indeed the case as demonstrated above. $\triangle$

Remark 13. When we work with *formal power series* rather than with *power series*, not only do we have *almost all* the intuitions and tools from power series available to us, but we also sidestep the convergence issues entirely. $\triangle$