Math 184

Lecture Notes Section 2.3 [*]

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NOTE: The notes is a summary for materials discussed in the class and is not supposed to substitute the textbook. In particular, the proofs here might omit some details for brevity, and are not supposed to be how you write proofs in the exam. Please refer back to the textbook when studying for exams; materials that appear in the textbook but do not appear in the lecture notes might still be tested. Please send me an email if you find typos.

1 Partition of an integer

Definition 1. Let n be a positive integer. A *partition* of an integer n is a sequence of positive integers $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ such that

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k \geq 1;$$

$$\lambda_1 + \lambda_2 + \ldots + \lambda_k = n.$$

That is to say, $a_1, \ldots, a_k$ is a sequence of weakly decreasing positive integers that sums up to n. We will often write $(\lambda_1, \ldots, \lambda_k) \vdash n$ as a shorthand for $(\lambda_1, \ldots, \lambda_k)$ is a partition of n. △

Example 2. The integer 4 can be partitioned in five distinct ways:

- (4);
- (3,1);
- (2,2);
- (2,1,1);
- (1,1,1,1). △

Remark 3. The partition of the integer n should not be confused with the partition of the set $[n] = \{1, \ldots, n\}$. The latter is a set $\{B_1, \ldots, B_k\}$ (unordered) of non-empty disjoint subsets of $[n]$ for which the union is equal to $[n]$. △

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Remark 4. The partition of an integer n is a special case of a (strong) composition of n. Recall that the latter is a sequence of integers $(a_1, \ldots, a_k)$ (ordered) such that

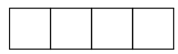
$$a_1, a_2, \ldots, a_k \geq 1;$$

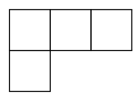
$$a_1 + a_2 + \ldots + a_k = n.$$

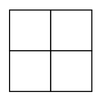
In particular, a composition $(a_1, \ldots, a_k)$ of n is a partition if and only if $a_1 \geq a_2 \geq \ldots \geq a_k$. $\triangle$

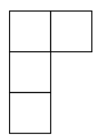
Definition 5 (Young diagram)**.** The *Young diagram* (Ferrer shape in the textbook) of the partition $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a diagram that consists of squares that form part of rectangular grid. The first row of the diagram consists of λ_1 squares, the second row λ_2 squares, and so on. $\triangle$

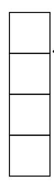
Example 6. The Young diagram for integer partitions of 4 are

$(4) \mapsto$ 

$(3,1) \mapsto$ 

$(2,2) \mapsto$ 

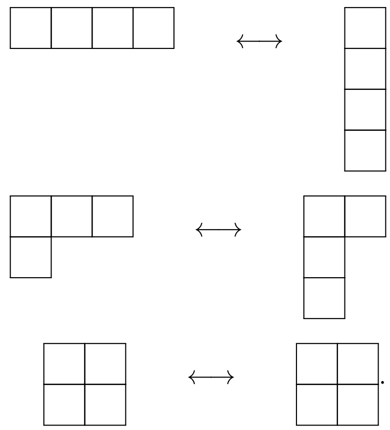
$(2,1,1) \mapsto$ 

$(1,1,1,1) \mapsto$ .

In particular, note that the total number of boxes in the Young diagrams above are always equal to 4 (as they correspond to integer partitions of 4). $\triangle$

Definition 7 (Conjugate)**.** The *conjugate* of a partition λ is the partition λ' such that the Young diagram of λ' is the reflected image of the Young diagram of λ through the main NW-SE diagonal. $\triangle$

Example 8. The following is a list of the Young diagram of integer partitions of 4 and their conjugates:



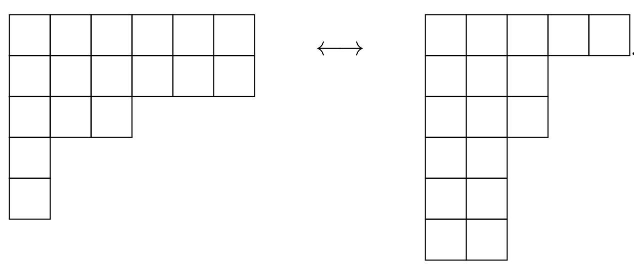
Note that taking conjugation twice sends a partition back to itself. $\triangle$

Proposition 9. *Let n be a positive integer. Consider the following two sets of partitions of n:*

$$S := \{(\lambda_1, \ldots, \lambda_k) \mid \lambda_1 = \lambda_2\}$$
$$T := \{(\lambda_1, \ldots, \lambda_k) \mid \lambda_1,\ \lambda_2,\ \ldots,\ \lambda_k \geq 2\}$$

Then $|S| = |T|$.

Proof (by picture). Note that the set S and T are conjugates of each other:



(This does not count as a rigorous proof.) $\square$

2 Historical notes

It is nigh impossible to talk about partitions of integers without mentioning the seminal work of Hardy and Ramanujan on the number of integer partitions.

Definition 10. For any integer n, we denote by $p(n)$ the number of integer partitions of n. $\triangle$

The following theorem was due to Hardy and Ramanujan (1918) and independently by Uspensky (1920).

Theorem 11. *We have the following asymptotic expression for $p(n)$:*

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right) \qquad \text{as } n \to \infty.$$

An exact formula for $p(n)$ exists, but they are highly complicated and are not easy to compute. In contrast, the formulas in Theorem 11 are more ubiquitous in practise because of the ease in computation (with a small error term as a trade-off).
Historically, Theorem 11 was a groundbreaking result when it was first published, as back then computing $p(n)$ for $n > 200$ was considered a Herculean task. More importantly though, is the method used in obtaining the result rather than the result itself. The proof involves a combination of combinatorics from 18th century, complex analysis from late 19th century, and number theory also late 19th century. In particular, some of the tools (such as Cauchy integral formula) was not available yet at the time of Euler (more on him below).

3 Euler's pentagonal number theorem

Leonhard Euler (18th-19th century) was perhaps the most prolific mathematican of all time, with more than 500 books and papers during his lifetime (even more impressive, he was blind for the last 20 years of his life!). He also has the honor of being the individual with the highest number of theorems/functions/equations named after him in the history of humanity (49 in wikipedia). In fact, This section is about one such theorem.

Definition 12. Let n be any positive integer. We write

$$\begin{aligned} P_d(n) &:= \{(\lambda_1, \ldots, \lambda_k) \vdash n \mid \lambda_1 > \lambda_2 > \ldots > \lambda_k\}; \\ p_d(n) &:= |P_d(n)|. \end{aligned}$$

That is to say, $p_d(n)$ is the number of partitions of n into *distinct* parts. We also write

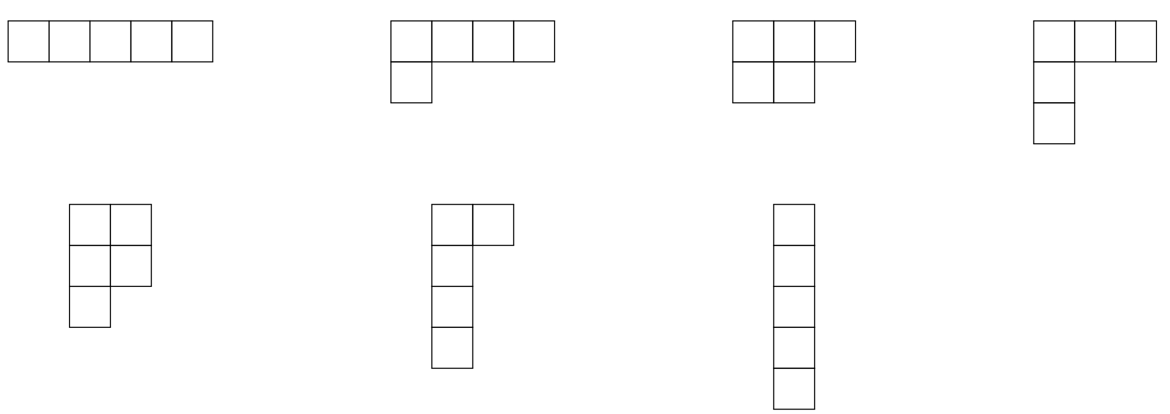
$$\begin{aligned} P_{d,odd}(n) &:= \{(\lambda_1, \ldots, \lambda_k) \vdash n \mid \lambda_1 > \lambda_2 > \ldots > \lambda_k;\ k \equiv 1 \bmod 2\}; \\ p_{d,odd}(n) &:= |P_{d,odd}(n)|. \end{aligned}$$

That is to say, $p_{d,odd}(n)$ is the number of partitions of n into *odd* number of *distinct* parts. Analogously, we write

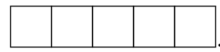
$$\begin{aligned} P_{d,even}(n) &:= \{(\lambda_1, \ldots, \lambda_k) \vdash n \mid \lambda_1 > \lambda_2 > \ldots > \lambda_k;\ k \equiv 0 \bmod 2\}; \\ p_{d,even}(n) &:= |P_{d,even}(n)|. \end{aligned}$$

That is to say, $p_{d,even}(n)$ is the number of partitions of n into *even* number of *distinct* parts. △

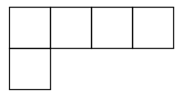
Example 13. Let $n = 5$. All the partitions of 5 are given by

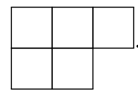


The partition of 5 into odd number of distinct parts are



The partition of 5 into even number of distinct parts are





In particular, we have in this case that $p_{d,odd}(5) - p_{d,even}(5) = -1$. △

Theorem 14 (Euler's pentagonal number theorem)**.** *Let n be a positive integer. Then*

$$p_{d,even}(n) - p_{d,odd}(n) = \begin{cases} (-1)^j & \text{if } n = \frac{j(3j-1)}{2} \text{ for } j = \pm 1, \pm 2, \pm 3, \ldots \\ 0 & \text{otherwise.} \end{cases}$$

Euler proved Theorem 14 by (heavy) algebraic manipulation, as the idea of bijective proof was not available/formalized yet during 18th century. We will give a modern proof of Theorem 14, which is (almost) bijective in nature.

Remark 15. The reason why Theorem 14 is called pentagonal number theorem is because the values of n for which $p_{d,odd}(n) - p_{d,even}(n) \neq 0$ are exactly the (generalized) pentagonal numbers,

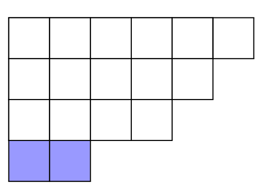
$$n = \frac{j(3j-1)}{2} \qquad j = \pm 1, \pm 2, \pm 3, \ldots.$$

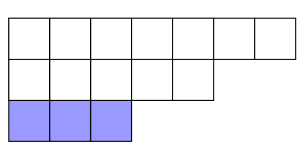
This is the number of distinct dots in a pattern of dots consisting of the outlines of regular pentagons with sides consisting of j dots (See picture in the lecture). △

We now build toward the proof of Theorem 14, which requires the following definition.

Definition 16. Let $\lambda := (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be an integer partition of n. We denote by $\mathrm{last}(\lambda)$ the number λ_k, which is the length of the shortest row of the Young diagram of λ. △

Example 17. Here are some examples of Young diagram λ with the corresponding value of $\mathrm{last}(\lambda)$:

$\lambda =$  has $\mathrm{last}(\lambda) = 2$;

$\lambda =$  has $\mathrm{last}(\lambda) = 3$;

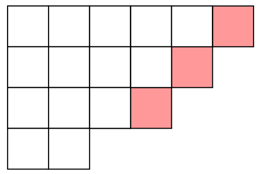
Note that the last row of the Young diagram are colored in blue. △

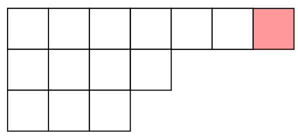
Definition 18. Let $\lambda := (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be an integer partition of n. We write

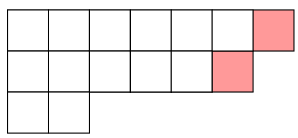
$$\text{stair}(\lambda) := \text{the largest integer } j \text{ so that } \lambda_1, \ldots, \lambda_j \text{ satisfies } \lambda_2 - \lambda_1 = \lambda_3 - \lambda_2 = \ldots = \lambda_j - \lambda_{j-1} = 1.$$

That is to say, stair(λ) is the the number of squares in the rightmost 45 degree line of the diagram (see the picture in the lecture]. The *stairs* of λ are the last square in the first, second, ..., stair(λ)-th row of λ. $\triangle$

Example 19. Here are some examples of Young diagram λ with the corresponding value of last(λ):

$\lambda =$  has stair(λ) = 3;

$\lambda =$  has stair(λ) = 1;

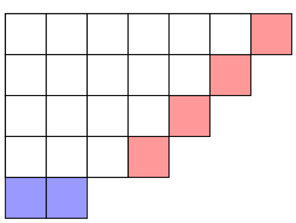
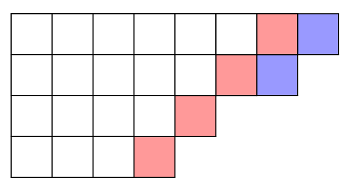
$\lambda =$  has stair(λ) = 2.

Note that the stairs of the Young diagram are colored in red. $\triangle$

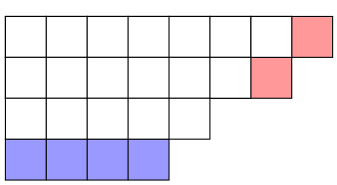
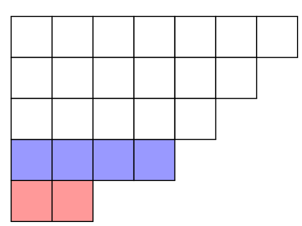
We use these new terminologies to define a transformation on Young diagrams with distinct part.

Definition 20. Let n be any positive integer. We define the function g that maps a Young diagram λ with distinct parts into a Young diagram $g(\lambda)$ by

- If last(λ) $\leq$ stair(λ), then remove the last row of λ and distribute its squares among the first last(λ) rows of λ (i.e., adding one square to each row), e.g.,

 transforms to 

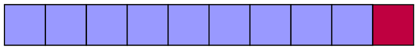
- If last(λ) $>$ stair(λ), then remove the stairs of λ and use these boxes to create a new last row of length stair(λ), e.g.,

 transforms to 

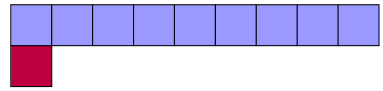
△

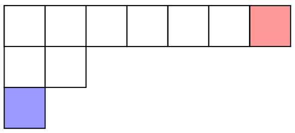
Remark 21. It follows from the construction of g that, if λ is a Young diagram with distinct parts, then $g(\lambda)$ is again a Young diagram with distinct parts, UNLESS when λ is a diagram from Example 26! △

Example 22. Here is the illustration of the function g when $n = 10$:



is associated with





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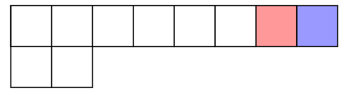

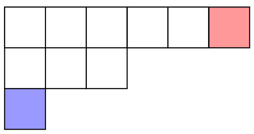

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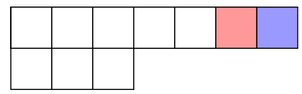

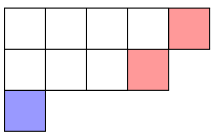

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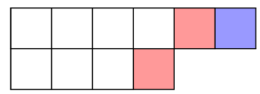

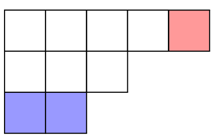

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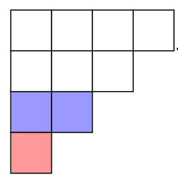
.

In particular, this shows that $p_{d,odd}(10) = p_{d,even}(10)$. △

Here are several important properties of the map g in Example 22:

- The function g maps partitions with odd parts to partitions with distinct parts, and vice versa;
- By composing the function g twice (i.e., the function $g \circ g$), we get the identity function.

In particular, the second property of g is important enough that it deserves its own name.

Definition 23 (Involution)**.** A function $g : X \to X$ on a finite set X is an *involution* if

$$g(g(x)) = x \quad \text{for all } x \in X,$$

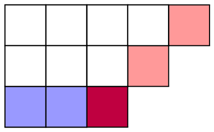
or equivalently if $g \circ g$ is the identity map on X. △

Exercise 24. Suppose that $g : X \to X$ is an involution. Show that

- the function g is a bijection on X;
- Let $\{X_1, X_2\}$ be a set partition of X such that $g(X_1) = X_2$. Then the function g (with domain restricted to X_1) is a bijection between X_1 and X_2. △

As a consequence of Exercise 24, we can conclude that $p_{d,odd}(n) = p_{d,even}(n)$ if g is an involution on $P_d(n)$ that maps $P_{d,odd}(n)$ to $P_{d,even}(n)$. This is true for most values of n (just like in Example 22), but there are two scenarios where things can go wrong, as outlined in the next example.

Example 25. Consider the partition λ of n for which $\text{last}(\lambda) = \text{stair}(\lambda)$, e.g.,



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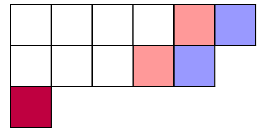


In this example, λ has 3 rows (odd), and yet $g(\lambda)$ still has 3 rows (odd). This means that g fails to map $P_{d,odd}(n)$ to $P_{d,even}(n)$.

It can be shown that (do it as an exercise) that this scenario (i.e., $\text{last}(\lambda) = \text{stair}(\lambda)$) happens only to the diagram

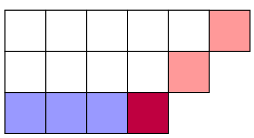
$$\lambda_{pos} := (2j-1, 2j-2, \ldots, j+1, j) \quad \text{for some } j \geq 1.$$

Note that λ_{pos} is a partition of the integer n given by

$$n = j + (j+1) + \ldots + (2j-2) + (2j-1) = \frac{j(3j-1)}{2}.$$

In particular, λ_{pos} does NOT exist if n is not a (positive) pentagonal number! △

Example 26. Consider the partition λ of n for which $\text{last}(\lambda) = \text{stair}(\lambda) + 1$, e.g.,



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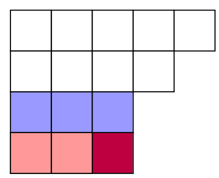

In this example, $g(\lambda)$ has two rows with same length (the third and fourth row). This means that g fails to map $P_d(n)$ to $P_d(n)$.

It can be shown that (do it as an exercise) that this scenario (i.e., $\text{last}(\lambda) = \text{stair}(\lambda) + 1$) happens only to the diagram

$$\lambda_{neg} := (2i-2, 2i-3, \ldots, i+1, i) \quad \text{for some } i \geq 2.$$

Note that λ_{neg} is a partition of the integer n given by

$$n = i + (i+1) + \ldots + (2i-3) + (2i-2) = \frac{(i-1)(3i-2)}{2}.$$

Substituting $j = 1 - i$, we have

$$n = \frac{((1-j)-1)(3(1-j)-2)}{2} = \frac{j(3j-1)}{2}, \qquad \text{for some } j \leq -1.$$

In particular, λ_{neg} does NOT exist if n is not a (negative) pentagonal number! △

Proposition 27. *Let n be any positive integer, and let g be as in Definition 20. Then*

- *The function g is an involution on the set $P_d(n) - \{\lambda_{pos}, \lambda_{neg}\}$;*

- *The function g maps $P_{d,odd}(n) - \{\lambda_{pos}, \lambda_{neg}\}$ to $P_{d,even}(n) - \{\lambda_{pos}, \lambda_{neg}\}$;*

- *λ_{pos} exists only if $n = \frac{j(3j-1)}{2}$ with $j \geq 1$. Analogously, λ_{neg} exists only if $n = \frac{j(3j-1)}{2}$ with $j \leq -1$;*

- *λ_{pos} is contained in $P_{odd}(n)$ if j is odd, and is contained in $P_{even}(n)$ if j is even. Analogously, λ_{neg} is contained in $P_{odd}(n)$ if j is odd, and is contained in $P_{even}(n)$ if j is even.*

Proof. This follows almost immediately from all the explanations we have so far. However, to test your understanding, write down a rigorous proof checking all claims made up to this point as an exercise. $\square$

Finally, Theorem 14 follows as a consequence of Proposition 27. This completes our proof of Theorem 14.

Exercise 28. Read Textbook Section 2.3, and familiarize yourself with other examples not presented in the lecture. $\triangle$