Math 184

Lecture Notes Section 2.2 *

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NOTE: The notes is a summary for materials discussed in the class and is not supposed to substitute the textbook. In particular, the proofs here might omit some details for brevity, and are not supposed to be how you write proofs in the exam. Please refer back to the textbook when studying for exams. Please send me an email if you find typos.

1 Partition of a set

Definition 1. A *partition* of a set X into k blocks is a set $\{B_1, B_2, \ldots, B_k\}$, where

- B_1, B_2, …, B_k are pairwise-disjoint non-empty subsets of X, i.e.;

$$B_i \neq \varnothing \quad \text{for all } i; \qquad B_i \cap B_j = \varnothing \quad \text{for distinct } i \text{ and } j.$$

- We have

$$B_1 \cup B_2 \ldots \cup B_k = X \qquad \triangle$$

Example 2. There are six partitions of $\{1, 2, 3, 4\}$ into three blocks, namely

- $\{\{1, 2\}, \{3\}, \{4\}\}$;
- $\{\{1, 3\}, \{2\}, \{4\}\}$;
- $\{\{1, 4\}, \{2\}, \{3\}\}$;
- $\{\{2, 3\}, \{1\}, \{4\}\}$;
- $\{\{2, 4\}, \{1\}, \{3\}\}$;
- $\{\{3, 4\}, \{1\}, \{2\}\}$. $\triangle$

Remark 3. It is important to remember the ordering does NOT matter with the set partition, e.g.

$$\{\{1, 2\}, \{3, 4\}\} = \{\{3, 4\}, \{1, 2\}\}. \qquad \triangle$$

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2 Stirling number of the second kind

Definition 4 (Stirling number of the second kind)**.** Let n and k be positive integers. Then the number of partitions of $\{1, \ldots, n\}$ into k blocks is denoted by

$$S(n, k) = \left\{ {n \atop k} \right\},$$

and is called a *Stirling number of the second kind.* $\triangle$

Theorem 5. *For any positive integer n,*

$$\left\{ {n \atop 2} \right\} = 2^{n-1} - 1.$$

Proof. We will instead count the number of ordered subsets (B_1, B_2) for which the corresponding (unordered) subsets $\{B_1, B_2\}$ is a partition of $\{1, \ldots, n\}$. Let k be the cardinality of B_1. Here are three important observations:

- The number k ranges from 1 to $n - 1$. Note that 0 and n are excluded as B_1 can neither be the empty set nor the whole set.
- The set B_2 is determined once B_1 is determined, as

$$B_2 = [n] - B_1.$$

- B_1 is any subset of $[n]$ with k elements, and therefore the number of choices for B_1 is

$$\binom{n}{k}.$$

By the three observations above, we conclude that the number of ordered subsets (B_1, B_2) for which $\{B_1, B_2\}$ is a partition of $[n]$ is given by

$$\sum_{k=1}^{n-1} \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k} - \binom{n}{0} - \binom{n}{n} = 2^n - 1 - 1 = 2^n - 2.$$

By the division principle, the number of partitions $\{B_1, B_2\}$ of n into two blocks is then

$$\frac{1}{2}\left(2^n - 2\right) = 2^{n-1} - 1,$$

which proves the theorem. $\square$

We now give a more general version of Theorem 5 for general values of k.

Theorem 6. *For any positive integer n and any nonnegative integer $k \leq n$,*

$$\left\{ {n \atop k} \right\} = \frac{1}{k!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n = \frac{1}{k!} \left(k^n - \binom{k}{1}(k-1)^n + \binom{k}{2}(k-2)^n - \binom{k}{3}(k-3)^n + \ldots \right).$$

We will prove Theorem 6 in Section 2.4 after we have inclusion-exclusion principle in our inventory. We nor present a very useful recurrence relation for Stirling number.

Theorem 7. *For any positive integer n and any positive integer $k \leq n$,*

$$\left\{ n \atop k \right\} = \left\{ n-1 \atop k-1 \right\} + k \left\{ n-1 \atop k \right\}.$$

Proof. We show that both sides count the same objects:

- By the definition, the left side of the equation counts the number of partitions of $[n]$ into k blocks;
- The first term of the right side of the equation counts the number of partitions of $[n]$ into k blocks *where the element n forms a block by itself*, i.e. partitions of the form

  $$\{B_1, B_2, \ldots, B_{k-1}, \{n\}\}.$$

  Clearly, it follows from the definition that this number is equal to

  $$\left\{ n-1 \atop k-1 \right\}.$$

- The second term of the right side of the equation counts the number of partitions of $[n]$ into k blocks *where the element n shares a block with some other elements.* To construct such partition, first partition $[n-1]$ into k blocks

  $$\{B_1, \ldots, B_k\} \subseteq [n-1],$$

  then choose one block B_i and add n into that block to get a partition that we want. By the product principle, the number of partitions of this form is equal to

  $$\left\{ n-1 \atop k \right\} \times k.$$

The theorem now follows by combining the three observations above. $\square$

The following triangle is the counterpart of Pascal's triangle for Stirling number of the second kind:

$$\begin{array}{lccccccccccccc}
n=0 & & & & & & & 1 & & & & & & \\
n=1 & & & & & & 0 & & 1 & & & & & \\
n=2 & & & & & 0 & & 1 & & 1 & & & & \\
n=3 & & & & 0 & & 1 & & 3 & & 1 & & & \\
n=4 & & & 0 & & 1 & & 7 & & 6 & & 1 & & \\
n=5 & & 0 & & 1 & & 15 & & 25 & & 10 & & 1 & \\
n=6 & 0 & & 1 & & 31 & & 85 & & 65 & & 15 & & 1
\end{array}$$

Exercise 8. Show that, for all positive integers n and $k \leq n$:

$$\left\{ n \atop k \right\} = \sum_{i=0}^{n} \binom{n}{i} \left\{ n-i \atop k-1 \right\}.$$

The proof can be found in Section 2.2 of the textbook. $\triangle$

3 Historical notes for Stirling's number of the second kind

The first person to compute the Stirling's number is James Stirling during 18th century (he was a contemporary of Isaac Newton). However, Stirling's motivation in Stirling's number did NOT come from combinatorics, but rather by Calculus! Indeed, the mathematicians of that period were fascinated by the study of series expansions. In particular, Stirling was interested in *Newton series*, which is a way to write a function $f(x)$ as the sum

$$\begin{aligned} f(z) =& \sum_{k=0}^{\infty} a_k z(z-1)(z-2)\dots(z-k+1) \\ =& a_0 + a_1 z + a_2 z(z-1) + a_3 z(z-1)(z-2) + \dots. \end{aligned}$$

In particular, Stirling wanted to know what is the Newton series expansion of the function $f(z) = z^n$. He managed to figure out that the Newton series expansion for z^n is

$$z^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} z(z-1)\dots(z-k+1),$$

i.e., $a_k = \left\{ {n \atop k} \right\}$. Furthermore, he presented a method to compute the constants $\left\{ {n \atop k} \right\}$, which is the recurrence formula in Theorem 7. However, Stirling (unfortunately) never figured out the explicit formula for Stirling number in Theorem 6.

Exercise 9 (Challenging)**.** Show that, for any positive integer n and any variable z,

$$z^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} z(z-1)\dots(z-k+1).$$

△

4 Bell number

Definition 10 (Bell number)**.** The number of all partitions of $[n]$ is denoted by

$$B(n) := \sum_{k=0}^{n} \left\{ {n \atop k} \right\},$$

and is called a *Bell number.* △

Remark 11. Unfortunately, there is no closed form formula for Bell number. However, we do have a very good method(s) to approximate the value of Bell numbers, one of which will be discussed in Section 3. △

Theorem 12. *Show that, for all nonnegative integers n,*

$$B(n+1) = \sum_{k=0}^{n} B(k) \binom{n}{k}.$$

Proof. We show that both sides count the same objects.

- By the definition, the left side of the equation counts the number of partitions of $[n+1]$;
- The right side of the equation counts the number of partitions of $[n+1]$ in the following manner. Let $\{B_1, B_2, B_3, \dots, \}$ be a partition of $[n+1]$ (number of blocks is arbitrary), and without loss of generality let B_1 be the block that contains $n+1$. Let h be the number of elements in B_1 that is contained in $[n]$. We make these two important observations:

- The number h ranges from 0 to n. Since B_1 is formed by adding $n+1$ into a subset of $[n]$ with h elements, the number of choices we have for B_1 is

$$\binom{n}{h}.$$

- The set $\{B_2, B_3, \ldots,\}$ is a partition of the set $[n] - B_1$, which has $n-h$ many elements. By the definition of Bell number, the number of choices we have for $\{B_2, B_3, \ldots\}$ is

$$B(n-h).$$

Combining the two observations above, we then conclude that

$$B(n+1) = \sum_{h=0}^{n} \binom{n}{h} B(n-h).$$

Substitute $h = n-k$, we get

$$B(n+1) = \sum_{k=0}^{n} \binom{n}{n-k} B(k) = \sum_{k=0}^{n} \binom{n}{k} B(k),$$

as desired. $\square$