Math 184

Lecture Notes Section 2.1 *

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NOTE: The notes is a summary for materials discussed in the class and is not supposed to substitute the textbook. In particular, the proofs here might omit some details for brevity, and are not supposed to be how you write proofs in the exam. Please refer back to the textbook when studying for exams. Please send me an email if you find typos.

1 Weak compositions

Example 1. Donald Duck has four candies to be shared among his three nephews Huey, Dewey, and Louie. How many distributions are possible for these four candies? △

Proof. Let a_1 be the number of candies assigned to Huey, a_2 the number of candies assigned to Dewey, and a_3 the number of candies assigned to Louie. It suffices to count the ordered 3-tuple (a_1, a_2, a_3) of nonnegative integers satisfying

$$a_1 + a_2 + a_3 = 4.$$

We would like to convert a tuple (a_1, a_2, a_3) into a string $b_1 b_2 \ldots b_6$ of length 6 with characters taken from the set $\{\square, \mathbf{1}\}$. The conversion follows the following rule:

- The number of $\mathbf{1}$'s in the string $b_1 b_2 \ldots b_6$ that appears before the first $\square$ symbol is equal to a_1;
- The number of $\mathbf{1}$'s in the string $b_1 b_2 \ldots b_6$ that appears after the first $\square$ symbol but before the second $\square$ symbol is equal to a_2;
- The number of $\mathbf{1}$'s in the string $b_1 b_2 \ldots b_6$ that appears after the second $\square$ symbol is equal to a_3.

Here are some examples of the ordered 3-tuple (a_1, a_2, a_3) and the corresponding string $b_1 b_2 \ldots b_6$.

$$\begin{array}{lcl@{\qquad\qquad}lcl}
(4,0,0) & \mapsto & \mathbf{1\,1\,1\,1}\square\square; & (3,1,0) & \mapsto & \mathbf{1\,1\,1}\square\mathbf{1}\square; \\
(3,0,1) & \mapsto & \mathbf{1\,1\,1}\square\square\mathbf{1}; & (2,2,0) & \mapsto & \mathbf{1\,1}\square\mathbf{1\,1}\square; \\
(2,1,1) & \mapsto & \mathbf{1\,1}\square\mathbf{1}\square\mathbf{1}; & (2,0,2) & \mapsto & \mathbf{1\,1}\square\square\mathbf{1\,1}.
\end{array}$$

Here are some properties that the string $b_1 b_2 \ldots b_6$ needs to satisfy:

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- There are exactly four **1**'s in the string $b_1 b_2 \dots b_6$. This is to match the number of candies;
- There are exactly 2=3-1 $\square$'s in the string $b_1 b_2 \dots b_6$. The first box acts as the boundary between **1**'s assigned to Huey and **1**'s assigned to Dewey; the second box acts as the boundary between **1**'s assigned to Dewey and **1**'s assigned to Louie.

It follows from the procedure outlined above that the 3-tuples (a_1, a_2, a_3) that we are counting is in bijection with the strings $b_1 b_2 \dots b_6$ satisfying the two conditions above. Finally, note that the number of such strings is equal to

$$\binom{6}{2} = 15,$$

as we need to choose exactly two elements in the string of six characters to be equal to the $\square$ symbol. Hence the answer to the problem is $\binom{6}{2} = 15$. $\square$

The objects that we count in the example above is a special case of weak compositions.

Definition 2. A *weak composition* of n into k parts is an ordered k-tuple $(a_1, a_2, \dots, a_k)$ of **nonnegative** integers satisfying

$$a_1 + a_2 + \ldots + a_k = n. \qquad \triangle$$

Theorem 3. *The number of weak compositions of n into k parts is*

$$\binom{n+k-1}{k-1} = \binom{n+k-1}{n}.$$

Proof. We convert an ordered tuple (a_1, a_2, a_3, a_n) into a string $b_1 b_2 \dots b_{n+k-1}$ of length $n + k - 1$ with characters taken from the set $\{\square, \mathbf{1}\}$ using the following rule:

- The number of **1**'s in the string that appears before the first $\square$ symbol is equal to a_1;
- The number of **1**'s in the string that appears after the first $\square$ symbol but before the second $\square$ symbol is equal to a_2;
- The number of **1**'s in the string that appears after the second $\square$ symbol but before the third $\square$ symbol is equal to a_3;
- $\dots$;
- The number of **1**'s in the string that appears after the $k - 1$-th $\square$ symbol is equal to a_k.

The results of this conversion are strings $b_1 b_2 \dots b_{n+k-1}$ satisfying

- There are exactly n many **1**'s in the string.
- There are exactly $k-1$ $\square$'s in the string $b_1 b_2 \dots b_6$, which are acting as the boundaries between strings of **1**'s that correspond to distinct a_i's.

We have learnt from Section 1.2 that the number of such strings is equal to

$$\binom{n+k-1}{k-1},$$

which completes the proof. $\square$

Exercise 4. Read Textbook Section 2.1.1 on the connection between multisets and weak compositions. $\triangle$

2 Compositions

We now give a slight twist to the scenario in Example 1.

Example 5. Donald Duck has four candies to be shared among his three nephews Huey, Dewey, and Louie. How many distributions are possible for these four candies, if *every nephew needs to get at least one candy*? $\triangle$

Proof. Donald shares the candies using the following strategy: He first gives one candy to each of the nephew, so he now has $1 = 4 - 3$ candy left. Then he shared the remaining 1 candies among his three nephews. By Theorem 3, Donald has

$$\binom{1+3-1}{3-1} = \binom{3}{2} = 3$$

ways to share the remaining candies. Hence we conclude that 3 is the answer to the problem. $\square$

The objects that we count above is a special case of (normal) composition.

Definition 6. A *composition* of n into k parts is an ordered k-tuple $(a_1, a_2, \ldots, a_k)$ of **positive** integers satisfying

$$a_1 + a_2 + \ldots + a_k = n.$$

$\triangle$

Remark 7. The difference between a weak composition and a composition is that, in the former case, the a_i's are **nonnegative** integers, while in the latter case, the a_i's are **positive** integers. $\triangle$

Theorem 8. *The number of compositions of n into k parts is*

$$\binom{n-1}{k-1}.$$

Proof. An ordered tuple $(a_1, a_2, \ldots, a_k)$ is a composition of n into k parts **if and only if** $(a_1 - 1, a_2 - 1, \ldots, a_k - 1)$ is a weak composition of $n-k$ into k parts. Indeed this is because

$$\begin{aligned} a_1 \geq 1;\ a_2 \geq 1;\ \ldots;\ a_k \geq 1 \quad &\iff \quad a_1 - 1 \geq 0;\ a_2 - 1 \geq 0;\ \ldots;\ a_k - 1 \geq 0; \\ a_1 + a_2 + \ldots + a_k = n \quad &\iff \quad (a_1 - 1) + (a_2 - 1) + \ldots + (a_k - 1) = n - k. \end{aligned}$$

Finally, note that by Theorem 3 the number of weak compositions of $n-k$ into k parts is equal to

$$\binom{(n-k)+k-1}{k-1} = \binom{n-1}{k-1},$$

and our proof is complete. $\square$