Math 184

Lecture Notes Section 1.5 *

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NOTE: The notes is a summary for materials discussed in the class and is not supposed to substitute the textbook. In particular, the proofs here might omit some details for brevity, and are not supposed to be how you write proofs in the exam. Please refer back to the textbook when studying for exams. Please send me an email if you find typos.

1 Pigeonhole principle

Example 1. There are at least two students in our class that share the same birthday month. △

Proof. We will use proof by contradiction. Suppose to the contrary that the claim is false.

Now note that there are twelve months:

January, February, March, April, May, June, July, August, September, October, November, December.

Under our assumption, there is at most one student with January birthday month, one student with February birthday month, etc. So our class can have at most 12 students. However, the last time I check, our class has 21 students, so this is a contradiction, and our proof is complete. □

The example above is a special case of the pigeonhole principle. (See figure in the lecture)

Theorem 2 (Pigeonhole principle)**.** *Let $A_1, A_2, \ldots, A_k$ be finite sets that are pairwise disjoint. Suppose that*

$$|A_1 \cup A_2 \cup \ldots \cup A_k| > kr.$$

There there exists at least one index i so that $|A_i| > r$.

Proof. Suppose to the contrary that the theorem is false. Then $|A_i| \leq r$ for all i, which implies that

$$\begin{aligned} |A_1 \cup A_2 \cup \ldots \cup A_k| =& |A_1| + |A_2| + \ldots + |A_k| \qquad \text{(by generalized addition principle)} \\ \leq& kr. \end{aligned}$$

This contradicts the assumption that $|A_1 \cup A_2 \cup \ldots \cup A_k| > kr$, and our proof is complete. □

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Remark 3. Pigeonhole principle tells you that there exists an i for which $|A_i| > r$, but it does NOT tell you which i! This is what mathematicians usually call a *non-constructive argument.* $\triangle$

Example 4. Consider the sequence

$$a_i := 2^i - 1, \qquad i \geq 1$$

i.e., the sequence $1, 3, 7, 15, 31, \ldots$ Then our sequence contains an elements that is divisible by 9. $\triangle$

Note that there is nothing special about 9; it can be replaced by any odd integer q.

Proof. Let r_i be the remainder of a_i after division by 9. Some examples:

$$a_1 = 1 \equiv 1 \bmod 9; \qquad r_1 = 1;$$

$$a_2 = 3 \equiv 3 \bmod 9; \qquad r_2 = 3;$$

$$a_3 = 7 \equiv 7 \bmod 9; \qquad r_3 = 7;$$

$$a_4 = 15 \equiv 6 \bmod 9; \qquad r_4 = 6.$$

Note that all r_i's are contained in $\{0, 1, 2, 3, \ldots, 8\}$ (why?). By pigeonhole principle there exists an index i and j with $i < j$ so that

$$r_i = r_j.$$

This implies that

$$(2^j - 1) - (2^i - 1) \equiv 0 \bmod 9.$$

Now note that the left side of the equation above is equal to

$$2^i(2^{j-i} - 1).$$

This implies that $2^i(2^{j-i} - 1)$ is divisible by 9. Since 9 is an odd number, we have 2^i is not divisible by 9. Hence we conclude that $2^{j-i} - 1$ is divisible by 9, which proves our claim. $\square$

Example 5. Show that, from a party of six people (Adams, Franklin, Hamilton, Madison, Jefferson, Washington), we can choose three people so that they either know each other, or they do not know each other. $\triangle$

Proof. Pick one person from the party of six people, say Washington. There are five other people in the party. By the pigeonhole principle, there are at least three people such that either (1) all those three people know Washington, or (2) all those three people do NOT know Washington. Without loss of generality, we will assume that these three people are Adams, Franklin, and Hamilton, and that all of them know Washington. Now note that

- If Adam and Franklin also knows each other, then Adam, Franklin, and Washington are the three people that knows each other, and we are done with the proof.

- If Adam and Hamilton also knows each other, then Adam, Hamilton, and Washington are the three people that knows each other, and we are done with the proof.

- If Franklin and Hamilton also knows each other, then Franklin, Hamilton, and Washington are the three people that knows each other, and we are done with the proof.

Then the only way for all these three scenarios above to fail if Adam, Franklin, and Hamilton do not know each other. In this case, these three people are the group that we need to prove the theorem. This completes our proof. □

Remark 6. Example 5 is a special case of Ramsey's problem:

> What is the minimum number of guests $R(m, n)$ that must be invited so that at least m will know each other or at least n will not know each other?

We have shown in Example 5 that $R(3, 3)$ is at most equal to 6; showing that $R(3, 3)$ is exactly equal to 6 is left as an exercise.

Computing the exact value for $R(m, n)$ (even for very small values of m and n, e.g. $m = n = 5$) is a longstanding open problem in combinatorics! The following is a quote of Paul Erdös to illustrate the difficulty of this problem:

> "Imagine an alien force, vastly more powerful than us, landing on Earth and demanding the value of $R(5, 5)$ or they will destroy our planet. In that case, we should marshal all our computers and all our mathematicians and attempt to find the value. But suppose, instead, that they ask for $R(6, 6)$. In that case, we should instead attempt to destroy the aliens."

Taking into account the difficulty of this problem, the friendly instructor is willing to give two extra credits for students who manage to compute the exact value of $R(7, 7)$. △

Exercise 7. Please read two other examples in the textbook (Example 1.47 and 1.48) for other applications of the pigeonhole principle. △