Math 184

Lecture Notes Section 1.4 *

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NOTE: The notes is a summary for materials discussed in the class and is not supposed to substitute the textbook. In particular, the proofs here might omit some details for brevity, and are not supposed to be how you write proofs in the exam. Please refer back to the textbook when studying for exams. Please send me an email if you find typos.

1 Bijective proofs

Example 1. In a fictional Manhattan, the streets form a square grid (see picture), and each street is one-way to the north or to the east. Count the number of ways to drive from the point (0,0) to (3,2). $\triangle$

Proof. We convert this question to a more familiar object: two-elements subsets of $\{1, 2, 3, 4, 5\}$. Our conversion rule is to map the trajectory of the driving path to the set of positions of North steps, e.g.,

$$\begin{aligned} \text{North, North, East, East, East} \quad &\mapsto \quad \{1, 2\}; \\ \text{East, North, East, North, East} \quad &\mapsto \quad \{2, 4\}; \\ \text{North, East, East, East, North} \quad &\mapsto \quad \{1, 5\}. \end{aligned}$$

This reduces the problem to counting the number of two-elements subsets of $\{1, 2, 3, 4, 5\}$, which we know from Section 1.3 is equal to

$$\binom{5}{2}.$$

This completes our proof. $\square$

The key observation in the proof is the fact that the number of northeastern lattice paths is equal to the number of subsets of two elements, and we prove this fact by a map that "shows" these two sets have the same cardinality. This map is called a bijection, which has the following rigorous definition. (Also check out the picture in the lecture.)

Definition 2. Let $f : S \to T$ be a map between two finite sets S, T.

- The map f is *injective* if distinct elements in S are mapped to distinct elements in T.
- The map f is *surjective* if every element in T is the image of some element in S.
- The map f is *bijective* if it is both injective and surjective.

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$\triangle$

Example 3. Let S be the set of subsets of $\{1, 2, \dots, n\}$ with exactly 2 elements, e.g.,

$$\{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 4\}, \dots.$$

Let T be the set of subsets of $\{1, 2, \dots, n\}$ with exactly $n - 2$ elements, e.g.,

$$\{1, 2, \dots, n-2\}, \{3, 4, \dots, n-1, n\}, \dots.$$

Show that $|S| = |T|$. $\triangle$

Proof (by counting). From what we have learned from Section 1.3,

$$|S| = \binom{n}{2} = \frac{n!}{2!(n-2)!}.$$

Again by what we have learned from Section 1.3,

$$|T| = \binom{n}{n-2} = \frac{n!}{(n-2)!2!}.$$

These two numbers are clearly the same number, as desired. $\square$

Proof by bijection. Consider the map $f : S \to T$ given by

$$X \quad \mapsto \quad \{1, 2, \dots, n\} - X.$$

Some examples:

$$\begin{aligned} \{1, 2\} \quad &\mapsto \quad \{3, 4, \dots, n\}; \\ \{1, 3\} \quad &\mapsto \quad \{2, 4, 5, \dots, n\}; \\ \{n-1, n\} \quad &\mapsto \quad \{1, 2, \dots, n-2\}. \end{aligned}$$

It is straightforward to check that f is a bijection (Do it as an exercise). Therefore, f is a bijection and we have $|S| = |T|$ as desired. $\square$

When showing that two sets S and T have the same cardinality, a bijective proof is usually preferable (though sometimes unobtainable) to direct counting proof because of several reasons:

- The formula for $|S|$ and $|T|$ can be too complicated to write down/compute;
- (Pak) Bijective proof usually demonstrates that one has achieved a "better understanding" on the structure of the underlying objects;
- (Stanley) When a bijective proof exists, it is usually more elegant than proofs by other means;
- Etc.

Remark 4. For any two finite sets S and T with $|S| = |T|$, there always exists "a" bijection, namely by

- Enumerating elements of S as $s_1, s_2, \dots, s_n$;

- Enumerating elements of T as $t_1, t_2, \ldots, t_n$;
- Construct the bijection f by mapping s_1 to t_1, s_2 to t_2, $\ldots$, s_n to t_n.

This mapping is a bijection, but is usually not considered "canonical/natural", as it depends on our "choice" of enumeration. When researchers are looking for bijective proofs, they are usually looking for "canonical bijection" and not just any other bijection. $\triangle$

Example 5. Consider the set $[n] := \{1, \ldots, n\}$ for any natural number $n \geq 1$. Consider the following two sets:

$$S := \text{the set of bijections } f : [n] \to [n];$$

$$T := \text{ the set of ordered } n\text{-tuples } (a_1, a_2, \ldots, a_n) \text{ of } [n] \text{ without repetition.}$$

Show that $|S| = |T| = n!$. $\triangle$

Proof. We already know that $|T| = n!$ from Section 1.2. Therefore it suffices to give a bijection between S and T. The bijection $f : S \to T$ in this case is given by

$$g \in S \quad \mapsto \quad (g(1), g(2), g(3), \ldots, g(n)) \in T.$$

Here is an example for $n = 3$. In this case both S and T contains 6 elements, and the "natural" bijection is given by

$$\begin{bmatrix} g(1) = 1; \\ g(2) = 2; \\ g(3) = 3. \end{bmatrix} \mapsto (1, 2, 3); \qquad \begin{bmatrix} g(1) = 1; \\ g(2) = 3; \\ g(3) = 2. \end{bmatrix} \mapsto (1, 3, 2); \qquad \begin{bmatrix} g(1) = 2; \\ g(2) = 1; \\ g(3) = 3. \end{bmatrix} \mapsto (2, 1, 3);$$

$$\begin{bmatrix} g(1) = 2; \\ g(2) = 3; \\ g(3) = 1. \end{bmatrix} \mapsto (2, 3, 1); \qquad \begin{bmatrix} g(1) = 3; \\ g(2) = 1; \\ g(3) = 2. \end{bmatrix} \mapsto (3, 1, 2); \qquad \begin{bmatrix} g(1) = 3; \\ g(2) = 2; \\ g(3) = 1. \end{bmatrix} \mapsto (3, 2, 1).$$

It is straightforward to check that this map is a bijection (do it yourself as an exercise), and the proof is complete. $\square$

Remark 6. Let X be a set with n elements. We consider the two sets as in Example 5:

$$S := \text{the set of bijections } f : X \to X;$$

$$T := \text{ the set of ordered } n\text{-tuples } (a_1, a_2, \ldots, a_n) \text{ of } X \text{ without repetition.}$$

One can again conclude that $|S| = |T| = n!$. However, in this case there exists no "natural bijection" between these two sets anymore!

- Intuitively this is because in order to apply the same bijection as in Example 5 to X, one needs to enumerate the elements of X as $x_1, x_2, \ldots, x_n$. This enumeration breaks the "naturality" of the bijection. Any other bijection one can come up with inevitably requires a similar enumeration step.

- Rigorously, one can prove the non-existence of a "natural" bijection by using *combinatorial species*, which requires category theory and is beyond the scope of this course. $\triangle$

Exercise 7. Please read textbook page 25-27 on Catalan numbers, with emphasis on the bijection between lattice paths and Standard Young tableau. $\triangle$

2 Properties of binomial coefficients

Let n be any positive integer, and let k be a positive integer for which $k \leq n$. We have seen in Example 3 that

$$\binom{n}{k} = \binom{n}{n-k}. \tag{1}$$

We will now prove other useful properties of binomial coefficients.

Theorem 8. *Let n be any positive integer. Then*

$$\sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = 2^n.$$

Proof. Recall the binomial theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

The claim in the theorem follows from the binomial theorem by substituting $x = y = 1$. $\square$

Theorem 9. *Let n and k be nonnegative integers so that $k < n$. Then*

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}.$$

Proof. Define

$$S := \text{the set of } k+1\text{-elements subsets of } \{1,2,\ldots,\text{n+1}\};$$

$$T_1 := \text{the set of } k \text{ elements subsets of } \{1,2,\ldots,\text{n}\};$$

$$T_1 := \text{the set of } k+1 \text{ elements subsets of } \{1,2,\ldots,\text{n}\}.$$

We know from Section 1.2 that

$$|T_1| = \binom{n}{k}; \qquad |T_2| = \binom{n}{k+1}; \qquad |S| = \binom{n+1}{k+1}. \tag{2}$$

This implies that, to prove the theorem, it suffices to show that

$$|T_1 \cup T_2| = |S|.$$

We will do so by giving a bijection between $T_1 \cup T_2$ and S.

Here is the bijection from $f : S \to T_1 \cup T_2$:

$$X \in S \quad \mapsto \quad X - \{n+1\} \ \in T_1 \cup T_2.$$

That is to say, $f(X)$ is obtained from X by removing $n+1$ from X. Note that $f(X)$ is contained in T_1 is X contains $n+1$ (and hence one element is removed), and is contained in T_2 if X does not contain $n+1$ (and hence nothing is removed).

Here are examples for $n=3$ and $k=1$:

$$\begin{aligned} \{1,2\} \in S \quad &\mapsto \quad \{1,2\} \in T_2; \\ \{1,3\} \in S \quad &\mapsto \quad \{1\} \in T_1; \\ \{2,3\} \in S \quad &\mapsto \quad \{2\} \in T_1. \end{aligned}$$

It is straightforward to check that this mapping is indeed a bijection (do it yourself as an exercise). This completes the proof. □

Theorem 10. *Let n be any positive integer. Then*

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2.$$

Proof. We will first us (1) to rewrite the equation in the theorem into

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k}.$$

We define, for any nonnegative integer k for which $k \leq n$:

$$\begin{aligned} S &:= \text{the set of } n\text{-element subsets of } \{1,2,\ldots,2n-1,2n\}; \\ A_k &:= \text{the set of } k\text{-element subsets of } \{1,2,\ldots,n\} \\ B_k &:= \text{the set of } k\text{-element subsets of } \{n+1,n+2,\ldots,2n\}. \end{aligned}$$

We know from Section 1.2 that

$$|S| = \binom{2n}{n}; \qquad |A_k| = \binom{n}{k}; \qquad |B_{n-k}| = \binom{n}{n-k}.$$

This implies that, to prove the theorem, it suffices to show that

$$|S| = \left| \bigcup_{k=0}^{n} A_k \times B_{n-k} \right|.$$

Here is the bijection:

$$X \in S \quad \mapsto \quad (X \cap [n], X - [n]) \;\in A_k \times B_k,$$

where k is the number of elements in S that are less than or equal to n. Here are some examples for $n = 2$:

$$\begin{aligned}
\{1,2\} \in S &\quad\mapsto\quad \Big(\{1,2\}, \{\ \}\Big) \in A_2 \times B_0; \\
\{1,3\} \in S &\quad\mapsto\quad \Big(\{1\}, \{3\}\Big) \in A_1 \times B_1; \\
\{1,4\} \in S &\quad\mapsto\quad \Big(\{1\}, \{4\}\Big) \in A_1 \times B_1; \\
\{2,3\} \in S &\quad\mapsto\quad \Big(\{2\}, \{3\}\Big) \in A_1 \times B_1; \\
\{2,4\} \in S &\quad\mapsto\quad \Big(\{2\}, \{4\}\Big) \in A_1 \times B_1; \\
\{3,4\} \in S &\quad\mapsto\quad \Big(\{\ \}, \{3,4\}\Big) \in A_0 \times B_2.
\end{aligned}$$

Checking that the map described above is a bijection is left as exercise. The proof is now complete. □

Theorem 11. *Let n be a positive integer. Then*

$$\sum_{k=1}^{n} k \binom{n}{k}^2 = n \binom{2n-1}{n-1}.$$

Proof. We again use (1) to rewrite the equation above into

$$\sum_{k=1}^{n} k \binom{n}{k} \binom{n}{n-k} = n \binom{2n-1}{n-1}.$$

We will prove that the left-side and the right-side of the equation above count the same object using different methods.

Consider the Friendly Instructor University, which is a university with $2n$ people consisting of n teaching assistants and n students. An Academic Conduct Committee of n members is formed out of the $2n$ people in the university, which is being led by a president of the committee (who must be a teaching assistant). We consider two different ways to create the committee and select a president out of these $2n$ people:

- The right-side of the equation starts with choosing the president from the teaching assistants (which has n choices), then choose the remaining $n-1$ members of the committee from the rest of the $2n-1$ people (which has $\binom{2n-1}{n-1}$ choices);
- The left-side equation starts with determining the number of teaching assistants that will be in the committee, denoted by k (note that k can range from 1 to n). Then, k teaching assistants is selected to be part of the committee (which has $\binom{n}{k}$ choices), and one teaching assistant is being selected as the president (which has k choices). Finally, $n-k$ students are chosen to complete the committee (which has $\binom{n}{n-k}$ choices).

As both sides count the same objects, they are equal, and our proof is complete. □

The method used in the proof of Theorem 11 called the *double counting* method. This method is usually used to prove that two expressions are equal by showing that they are the cardinality of the same object counted in a different manner.

Exercise 12. Please read Textbook Section 1.4.3 on multinomial coefficients. $\triangle$