Math 184

Lecture Notes Section 1.3 *

Instructor: Swee Hong Chan

NOTE: The notes is a summary for materials discussed in the class and is not supposed to substitute the textbook. In particular, the proofs here might omit some details for brevity, and are not supposed to be how you write proofs in the exam. Please refer back to the textbook when studying for exams. Please send me an email if you find typos.

1 Division principle

Example 1. Five knights of King Arthur[1], namely Lancelot, Mordred, Gawain, Tristan, and Agravain are to be seated in a round table with five seats. Count the number of different seating arrangements for these knights. △

Answer. If the table were linear instead of circular (see picture in the lecture), then each seating arrangement is a permutation of the set of these five knights. Let T be

$$T := \text{the set of linear seating arrangements of these five knights.}$$

Then we have $|T| = 5!$ by materials from Section 1.2.

Now let S be

$$S := \text{the set of circular seating arrangements of these five knights.}$$

Each element of S corresponds to exactly 5 distinct elements of T. Indeed, for example, these five linear seating arrangements in T correspond to the same circular arrangement (drawn in the lecture):

AGLMT, GLMTA, LMTAG, MTAGL, TAGLM.

Hence we have

$$|T| = 5 \times |S|,$$

which then implies that

$$|S| = \frac{|T|}{5} = \frac{5!}{5} = \frac{120}{5} = 24.$$

Hence the answer to the problem is 24. □

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[1]also known as Altria, Arturia, or Artoria

Definition 2. Let S and T be finite sets, and let d be a fixed positive integer. We say that a function $f : T \to S$ is d-to-one if for each element $s \in S$ there exist exactly d elements $t \in T$ so that $f(t) = s$. (See the illustration in the lecture.) $\triangle$

An example of a d-to-one function appears in Example 1 with $d = 5$.

Theorem 3 (Division principle)**.** *Let S and T be finite sets so that a d-to-one function $f : T \to S$ exists. Then*

$$|S| = \frac{|T|}{d}.$$

Proof. This follows directly from the definition of d-to-one functions. $\square$

Example 4. Find the number of different seating arrangements for n people around a circular table for n people. $\triangle$

Answer. Let T be the set of all linear seating arrangements, and let S be the set of all circular seating arrangements. We will show that there exists an n-to-one function f from T to S.

For any linear seating arrangement $(a_1, a_2, \ldots, a_n)$ in T, the map f sends this linear seating arrangement to the circular seating arrangement $\langle a_1, \ldots, a_n \rangle$ in S (see the picture in the lecture).

This mapping is clearly n-to-one, as there are exactly n linear seating arrangements in T that is mapped to the circular seating arrangement $\langle a_1, \ldots, a_n \rangle$ in S, namely

$$\begin{aligned}
&(a_1, a_2, a_3, \ldots, a_{n-1}, a_n);\\
&(a_2, a_3, \ldots, a_{n-1}, a_n, a_1);\\
&(a_3, \ldots, a_{n-1}, a_n, a_1, a_2);\\
&\vdots\\
&(a_n, a_1, a_2, a_3, \ldots, a_{n-1}).
\end{aligned}$$

By the division principle, it then follows that our answer is

$$|S| = \frac{|T|}{n} = \frac{n!}{n} = \frac{n \times (n-1)!}{n} = (n-1)!,$$

which answers the problem. $\square$

2 Number of subsets inside a set

Example 5. A group of enemies[2] are attacking Camelot, and King Arthur needs to dispatch three knights to face the enemies out of the five knights currently available (the order of dispatch does NOT matter). How many different choices does King Arthur have? $\triangle$

[2] consisting of a modern Japanese teenager, a shield-carrying maiden, and a Renaissance painter

A *wrong* answer: King Arthur has 5 choices for the first knight, 4 choices for the second knight, and 3 choices for the third knight. Therefore, the number of all choices are $5 \times 4 \times 3$.
This is a wrong solution because the same triple of knights are counted many times. Indeed, suppose that L, G, and T are the three knights chosen. The above (wrong) solution considers L,G, T and G,L,T as *different* triples, whereas in the context of Example 5 they are considered *identical*.

The right answer to Example 5. Let S be the set of choices of three knights where order does not matter, and let T be the set of choices of three knights where order does matter.

First note that T is exactly the set of 3-lists of $\{L, G, T, M, A\}$ without repetition. In particular, we learn from Section 1.2 that

$$|T| = \frac{5!}{2!} = 60.$$

Let $f : T \to S$ be the function

$$f : T \to S$$
$$(a_1, a_2, a_3) \mapsto \{a_1, \ldots, a_3\}.$$

Note that f is a six-to-one function, by the correspondence

$$\{a_1, a_2, a_3\} \mapsto \begin{matrix} (a_1, a_2, a_3), & (a_1, a_3, a_2), & (a_2, a_1, a_3), \\ (a_2, a_3, a_1), & (a_3, a_1, a_2), & (a_3, a_2, a_1) \end{matrix}.$$

It then follows by the division principle that

$$|S| = \frac{|T|}{6} = \frac{60}{6} = 10.$$

Therefore the answer to the problem is 10 choices. $\square$

Example 5 is a special case of the following theorem.

Theorem 6. *Let X be a set of cardinality n, and let k be a positive integer such that $k \leq n$. Then the number of all k-element subsets of X*

$$\frac{n!}{(n-k)!k!}.$$

Proof. Let S be the set of all k-element subsets of X, and let T be the set of all k-lists of X without repetition. It follows from what we learn in Section 1.2 that

$$|T| = \frac{n!}{(n-k)!}.$$

Let $f : T \to S$ be the function

$$f : T \to S$$
$$(a_1, \ldots, a_k) \mapsto \{a_1, \ldots, a_k\}.$$

For each $\{a_1, a_2, \ldots, a_k\}$ in S, we learnt from Section 1.2 (i.e., when we learnt about permutations) there are exactly $k!$ elements in T that are mapped to S. Therefore f is a $k!$-to-one function, and the division principle then gives us

$$|S| = \frac{|T|}{k!} = \frac{n!}{(n-k)!k!},$$

which completes our proof. □

The number that appears in Theorem 6 is important enough that it has its own name and notation.

Definition 7 (Binomial coefficient)**.** For any positive integer n and any positive integer k for which $k \leq n$, the corresponding *binomial coefficient* is

$$\binom{n}{k} = \frac{n!}{(n-k)!k!},$$

also read as "n-choose-k". △

The reason why it is called a binomial coefficient is because of the following theorem.

Theorem 8 (Binomial theorem)**.** *If n is a positive integer, then*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Proof. The proof of this theorem is assigned as a homework problem. Check the textbook for a proof. □

Example 9. We compute the first few expansion of $(x+y)^n$ for small n:

$$\begin{aligned}
(x+y)^2 &= \binom{2}{2}x^2 + \binom{2}{1}xy + \binom{2}{0}y^2 = x^2 + 2xy + y^2; \\
(x+y)^3 &= \binom{3}{3}x^3 + \binom{3}{2}x^2y + \binom{3}{1}xy^2 + \binom{3}{0}y^3 = x^3 + 3x^2y + 3xy^2 + y^3; \\
(x+y)^4 &= \binom{4}{4}x^4 + \binom{4}{3}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{1}xy^3 + \binom{4}{0}y^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.
\end{aligned}$$

△

Remark 10 (This won't be tested)**.** The binomial coefficient has the following asymptotic formula when n and k are sufficiently large:

$$\binom{n}{k} \sim \exp\left[nH\left(\frac{k}{n}\right)\right],$$

where $H(p) := -p\log p - (1-p)\log(1-p)$ is called the binary entropy function. This approximation is very useful for research in probability and computer science; this approximation can be derived from Stirling's formula. △