Math 184

Lecture Notes Section 1.2 *

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NOTE: The notes is a summary for materials discussed in the class and is not supposed to substitute the textbook. In particular, the proofs here might omit some details for brevity, and are not supposed to be how you write proofs in the exam. Please refer back to the textbook when studying for exams. Please send me an email if you find typos.

1 Product Principle

Example 1. Count the number of positive integers with exactly two digits. △

Answer. Denote the first digit by a_1, and the second digit by a_2. We have that a_1 can be any element from $\{1, 2, \ldots, 9\}$, while a_2 can be any element from $\{0, 1, 2, \ldots, 9\}$. This means that

- There are 9 choices for a_1;
- For each choice of a_i, we have ten different choices for a_2;
- Therefore, in total we have $9 \times 10 = 90$ such integers.

This answers the problem. □

Definition 2 (Set products)**.** Let X and Y be two finite sets. The *set product* $X \times Y$ is the set

$$X \times Y := \{(x, y) \mid x \in X, y \in Y\}.$$

That is, the set $X \times Y$ consists of pairs (x, y) satisfying $x \in X$ and $y \in Y$. △

Theorem 3. *Let X and Y be finite sets. The cardinality of $X \times Y$ is given by*

$$|X \times Y| = |X| \times |Y|.$$

Proof. There are $|X|$ choices for the first element x of the pair (x, y). For each choice of x, there are $|Y|$ choices for y. Each choice of x can be paired with a choice of Y, so the theorem follows. □

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2 Generalized product principle

There is nothing special about product of *two* sets. Just like with addition principle, we also have generalized product principle.

Definition 4. Let $X_1, X_2, \ldots, X_k$ be finite sets. Then the *set product* $X_1 \times X_2 \times \ldots \times X_k$ is

$$X_1 \times X_2 \times \ldots \times X_k := \{(x_1, x_2, \ldots, x_k) \mid x_i \in X_i \text{ for } i = 1, 2, \ldots, k\}.$$

That is, the set $X_1 \times X_2 \times \ldots \times X_k$ consists of pairs $(x_1, x_2, \ldots, x_k)$ satisfying $x_1 \in X_1$, $x_2 \in X_2$, $\ldots$, and $x_k \in X_k$. $\triangle$

Theorem 5 (Generalized product principle)**.** *The cardinality of the set $X_1 \times X_2 \ldots \times X_k$ is*

$$|X_1 \times X_2 \times \ldots \times X_k| = |X_1| \times |X_2| \times \ldots \times |X_k|.$$

Example 6. Your friendly instructor is buying a new license plate for his new bicycle. The license plate needs to have six characters, the first two characters must be letters, while the next 4 characters must be digits. How many choices of license plates does your friendly instructor have? $\triangle$

Proof. Let $a_1a_2a_3a_4a_5a_6$ be the license plate of your friendly instructor. Note that

- a_1 and a_2 are chosen from the set $\{a, b, c, d, \ldots, x, y, z\}$.
- a_3, a_4, a_5, and a_6 are chosen from the set $\{0, 1, \ldots, 9\}$.

By the generalized product principle, the answer to this question is equal to

$$26 \times 26 \times 10 \times 10 \times 10 \times 10 = 6,760,000.$$

There are indeed many choices for your friendly instructor to pick. $\square$

The generalized product principle can be proved by a direct counting argument (try to do it yourself as an exercise!). However, for pedagogical reason, we will present a proof by using mathematical induction.

3 Mathematical induction

Theorem 7 (Mathematical induction)**.** *Suppose that you want to prove a statement $S(n)$ involving a number n holds for all values of $n \geq 1$ (or $n \geq 0$). You can give a proof by following these two steps:*

- *The **initial case** or **base case**: prove that $S(n)$ holds for $n = 1$ (or $n = 0$);*
- *The **induction step**, **inductive step**, or **step case**: prove that for every n, if the statement holds for n, then it holds for $n + 1$.*

If one can verify the two steps above, then the statement $S(n)$ holds for all $n \geq 1$ (or $n \geq 0$).

Example 8. Let $n \geq 1$. Show that the cardinality of the set $[n] = \{1, \ldots, n\}$ is

$$\big|\{1, \ldots, n\}\big| = n. \qquad \triangle$$

Proof. This statement can be proved by a direct counting argument, but let's prove it by mathematical induction.

- Base case: When $n = 1$, we have

  $$|\{1\}| = 1,$$

  by definition.

- Induction step: Suppose that the statement holds for n. We want to show that it also holds for $n + 1$. This can be done by

  $$\begin{aligned} |\{1, \ldots, n, n+1\}| =& |\{1, \ldots, n\}| + |\{n+1\}| \qquad \text{(by the addition principle)} \\ =& n + 1 \qquad \text{(by the induction assumption).} \end{aligned}$$

We have shown that the two steps of the induction hold. This proves the statement for all $n \geq 1$. □

We now go back to the proof of Theorem 5.

Proof of Theorem 5. We will use induction on the index k of the product $X_1 \times X_2 \ldots \times X_k$.

- Base case: When $k = 1$, the statement is tautologically true.

- Inductive step: Suppose that the statement holds for k. We will show that it also holds for $k + 1$. This can be done by

  $$\begin{aligned} |X_1 \times X_2 \times \ldots \times X_k \times X_{k+1}| =& |X_1 \times X_2 \times \ldots \times X_k| \times |X_{k+1}| \qquad \text{(by the product principle)} \\ =& |X_1| \times |X_2| \times \ldots \times |X_k| \times |X_{k+1}| \qquad \text{(by the induction assumption).} \end{aligned}$$

This completes our proof by mathematical induction, as desired. □

4 Counting permutations

Example 9. A rat, ox, tiger, rabbit, dragon, snake, horse, goat, monkey, rooster, dog, pig (twelve Chinese zodiacs) participate in a marathon race. There are no ties. In how many different ways can the competition end? △

Notice that we cannot apply the generalized counting principle directly to this example because our choices are not *independent* of each other. For instance, if the rat wins the race, it cannot finish third, fourth, or fifth. Hence another approach is needed.

Answer to Example 9. Let us solve this question by following these 12 steps:

- First of all, note that there are 12 choices for the winners.
- After we decode who the winner is, we can decide who the runner-up is. There are 12-1=11 choices (regardless of who wins), since the winner cannot be the runner-up.
- After we decide on the winner and the runner-up, there are 12-2=10 choices for the third place, since the winner and the runner-up cannot be third place.
- ...
- After we decide on the first eleven positions, there are 12-11=1 choice left for the twelfth position, as all other runners have been chosen by now.

Hence we have 12 choices for the winner, 11 choices for the runner-up, 10 choices for the third place, ..., 1 choice for the twelfth-place. By the generalized product principle, we conclude that the answer to this problem is

$$12 \times 11 \times 10 \times \ldots \times 2 \times 1 = 479,001,600,$$

which completes our proof. □

Remark 10. The key idea in the proof of Example 9 is that, while the *set* of possible choices for the i-th position depended on the choices we made previously, the *number* of these choices is always $13 - i$ (regardless of previously made choices). △

Example 9 is a special case of what we call a permutation of a set.

Definition 11 (Permutation)**.** A *permutation* of a finite set S is an ordered tuple $(a_1, \ldots, a_n)$ such that each element of S occurs exactly once (Note that by definition n must be the cardinality of S). A permutation is also known as a *list without repetition.* △

Definition 12. Let n be a nonnegative integer. The n-factorial is

$$n! := n \times (n-1) \times \ldots \times 2 \times 1,$$

the product of the first n positive integers. We adopt the convention that $0! = 1$. △

Theorem 13. *Let S be a finite set of cardinality n. The number of permutations of the set S is equal to $n!$.*

Proof. Let $(a_1, \ldots, a_n)$ be a permutation of S. Then

- a_1 can be any element of the set S, so there are n choices for a_1.
- After we choose a_1, we now have a_2 can be any element of the set $S - \{a_1\}$, so there are $n-1$ choices for a_2.
- After we choose a_1, a_2, we now have a_3 can be any element of the set $S - \{a_1, a_2\}$, so there are $n-2$ choices for a_3.

- For any i, after we choose $a_1, a_2, \ldots, a_{i-1}$, we now have a_i can be any element of the set $S - \{a_1, \ldots, a_{i-1}\}$, so there are $n+1-i$ choices for a_i.

Hence we have n choices for a_1, we have $n-1$ choices for a_2, we have $n-2$ choices for a_3, ..., we have 1 choice for a_n. By the generalized product principle, we conclude that the number of permutations of the set S is equal to

$$n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1 = n!,$$

as desired. □

Remark 14 (Stirling's approximation, this won't be tested)**.** The number $n!$ has the asymptotic formula

$$n! \;\sim\; \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

That is to say,

$$\lim_{n\to\infty} n! \times \left[\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\right]^{-1} = 1.$$

If you end up doing research on combinatorics and/or probability in the future, you will probably (and by that, I mean definitely) find this formula to be very useful. △

5 Counting partial lists without repetition

Example 15. Let's go back to the marathon race of the twelve zodiacs. Assume that the runners who arrive first, second, and third, will receive gold, silver, and bronze medal, respectively. How many different possibilities are there for the list of medal winners? △

Proof. We can use the argument from before:

- The gold medalist can be any one of the twelve zodiacs, so there are 12 choices for the gold medalist.
- After choosing the gold medalist, the silver medalist can be any one of the remaining eleven zodiacs, so there are 11 choices for the silver medalist.
- After choosing the gold and silver medalist, the bronze medalist can be one of the remaining ten zodiacs, so there are 10 choices for the bronze medalist.

By the generalized product principle, the answer to this problem is then

$$12 \times 11 \times 10 = 1320,$$

and our proof is complete. □

Definition 16. Let S be a finite set of cardinality n, and let k be a positive integer so that $k \leq n$. A k-list of the set S without repetition is an ordered tuple $(a_1, \ldots, a_k)$ of length k of elements of S so that $a_1, \ldots, a_k$ are pairwise distinct. △

Note that the objects counted in Example 15 is the set of 3-lists of S without repetition. Also note that a permutation of S is exactly an n-list of S without repetition.

Theorem 17. *Let S be a finite set of cardinality n, and let k be a positive integer so that $k \leq n$. The number of k-lists of S without repetition is equal to*

$$n \times (n-1) \times \ldots \times (n-k+1).$$

Proof. Let $(a_1, \ldots, a_k)$ be a k-list of S without repetition. We can use the argument from before:

- a_1 can be any element of the set S, so there are n choices for a_1.
- After we choose a_1, we now have a_2 can be any element of the set $S - \{a_1\}$, so there are $n-1$ choices for a_2.
- $\ldots$
- After we choose $a_1, a_2, \ldots, a_{k-1}$, we now have a_k can be any element of the set $S - \{a_1, \ldots, a_{k-1}\}$, so there are $n-k+1$ choices for a_k.

The conclusion of the theorem now follows from the generalized product principle. $\square$

The number that appears in Theorem 17 has the following more compact form.

Definition 18. For any positive integer n and any positive integer k for which $k \leq n$, we write

$$\begin{aligned}(n)_k := n \times (n-1) \times \ldots \times (n-k+1) &= \frac{n \times (n-1) \times \ldots \times (n-k+1) \times (n-k) \times (n-k-1) \times \ldots \times 1}{(n-k) \times (n-k-1) \times \ldots \times 1} \\ &= \frac{n!}{(n-k)!}.\end{aligned}$$

$\triangle$