Math 184

Lecture Notes Section 1.1 *

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NOTE: The notes is a summary for materials discussed in the class and is not supposed to substitute the textbook (which contains all the details not included here for brevity). Please refer back to the textbook when studying for exams. Please send me an email if you find typos.

1 Cardinality of a set

Definition 1 (Cardinality of a set)**.** Let X be a finite set. The *cardinality* of X, denoted by $|X|$, is the number of elements in the set X. △

Example 2. Let X be the set

$$X := \{\text{red}, \text{blue}, \text{black}, \text{yellow}, \text{pink}\}.$$

Then the cardinality of X is

$$|X| = 5,$$

because X has 5 elements. △

Example 3. Let X be the empty set, i.e.,

$$X := \varnothing = \{\ \ \}.$$

Then the cardinality of X is

$$|X| = 0,$$

since X does not contain anything. △

Example 4. Let n be any nonnegative integer, and let X be the set

$$X := [n] := \{1, 2, 3, \ldots, n\}.$$

Then the cardinality of X is

$$|X| = n,$$

since X contains n elements. △

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2 Addition principle

Recall that two sets are *disjoint* if they have no elements in common.

Theorem 5 (Addition principle)**.** *Let A and B be two finite disjoint sets. Then*

$$|A \cup B| = |A| + |B|. \tag{1}$$

Example 6. Let A and B be the set

$$A := \{\text{red}, \text{blue}, \text{black}, \text{yellow}, \text{pink}\};$$
$$B := \{\text{green}, \text{white}, \text{gold}\}.$$

The union between A and B is then

$$A \cup B = \{\text{red}, \text{blue}, \text{black}, \text{yellow}, \text{pink}, \text{green}, \text{white}, \text{gold}\}.$$

We can count how many elements in $A \cup B$ directly to conclude that

$$|A \cup B| = 8.$$

But we can also use addition principle since A and B are disjoint sets to get

$$|A \cup B| = |A| + |B| = 5 + 3 = 8,$$

which gives us the same conclusion. △

Addition principle (Theorem 5) seems to be obviously true, but since this is a proof-based course, we will include a short proof.

Proof of Theorem 5. Both sides of (1) counts the elements of the set $A \cup B$, as is argued below:

- The left side of (1) counts the number of elements in $A \cup B$ directly *by definition*.
- The right side of (1) counts the number of elements in $A \cup B$ by first counting all elements that comes from the set A, then followed by counting the elements that comes from the set B. Note that in this case, each element is counted *exactly once* (as A and B are disjoint).

So the two sides indeed count the same thing, as desired. □

Non-example 7. Let C be the set

$$C = \{\text{red}, \text{green}, \text{white}, \text{gold}\}.$$

Then union between A and C is then

$$A \cup C = \{\text{red}, \text{blue}, \text{black}, \text{yellow}, \text{pink}, \text{green}, \text{white}, \text{gold}\}.$$

Direct counting gives us

$$|A \cup C| = 8.$$

However, the right side of the addition principle is

$$|A| + |C| = 5 + 4 = 9,$$

so the two sides of the addition principle is not equal here! This is because the sets A and C are not disjoint (why?), and therefore the addition principle does not apply. $\triangle$

3 Generalized addition principle

The addition principle in Theorem 5 was about *two* finite sets, but there is nothing magical about number *two* here. Indeed, there is a generalized addition principle for any number of sets.

Theorem 8 (Generalized addition principle)**.** *Let $A_1, A_2, \ldots, A_n$ ($n \geq 2$) be finite sets that are pairwise disjoint. Then*

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = |A_1| + |A_2| + \ldots + |A_n|.$$

Example 9. Let A_1, A_2, A_3 be the set

$$\begin{aligned} A_1 :=& \{1, 6, 20\}; \\ A_2 :=& \{8, 17\}; \\ A_3 :=& \{12, 16, 33, 41\}. \end{aligned}$$

By the addition principle, the cardinality of $A_1 \cup A_2 \cup A_3$ is then equal to

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| = 3 + 2 + 4 = 9.$$

which is indeed correct. $\triangle$

Proof of Theorem 8. Again, both sides count the elements of the same set $A_1 \cup A_2 \cup \ldots \cup A_n$:

- The left side counts the number of elements in $A_1 \cup A_2 \cup \ldots \cup A_n$ directly *by definition.*
- The right side counts the number of elements in $A_1 \cup A_2 \cup \ldots \cup A_n$ by first counting all elements from A_1, then all elements from A_2, and so on until it counts all elements from A_n. Note that in this case, each element is counted *exactly once* (as the sets $A_1, \ldots, A_n$ are pairwise disjoint).

Therefore, they have to be equal. $\square$

4 Subtraction principle

Definition 10 (Set difference)**.** Let A and B be two finite sets. Their *set difference*, denoted by $A - B$, is the set consisting of elements of A that are not elements of B. $\triangle$

Example 11. Let A and B be the sets

$$A := \{\text{Washington, Jefferson, Roosevelt, Lincoln}\};$$
$$B := \{\text{Adams, Jefferson, Franklin, Sherman, Livingston}\}.$$

The set $A - B$ is then equal to

$$A - B = \{\text{Washington, Roosevelt, Lincoln}\},$$

by definition. △

Theorem 12 (Subtraction Principle)**.** *Let A be a finite set, and let $B \subseteq A$. Then*

$$|A - B| = |A| - |B|.$$

Example 13. Let A and B be the set

$$A := \{\text{Washington, Hamilton, Arnold}\};$$
$$B := \{\text{Arnold}\}.$$

By the subtraction principle, the cardinality of $A - B$ is equal to

$$|A - B| = |A| - |B| = 3 - 1 = 2.$$

This is indeed the cardinality of $A - B$, as can be verified by direct counting. △

Problem 14. Show that the subtraction principle does not apply to Example 11, and give a reason why. △

Proof of Theorem 12. We will first prove the equation

$$|A - B| + |B| = |A|. \tag{2}$$

Note that (2) is true by the addition principle. Indeed, the sets $A - B$ and B are disjoint by definition, and their union is equal to A by the assumption that $B \subseteq A$. The claim of Theorem 12 now follows by subtracting $|B|$ from both sides of (2). □

Example 15. What is the number of positive integers less than or equal to 1000 that have at least two different digits? △

Answer to Example 15. Let A and B the set

$$A := [1000] = \{1, 2, \ldots, 999, 1000\};$$
$$B := \{m \in A \mid \text{all digits in } m \text{ are equal}\}.$$

Note that the answer to Example 15 is exactly $|A - B|$ by definition. Also note that

$$|A| = 1000 \qquad \text{(by direct counting)};$$

and

$$\begin{aligned} B =& \{1, 2, \dots 9\} \cup \{11, 22, \dots, 99\} \cup \{111, 222, \dots, 999\} \qquad \text{(by definition)}; \\ |B| =& 9 + 9 + 9 \qquad \text{(by the addition principle)} \\ =& 27. \end{aligned}$$

Therefore, the answer to Example 15 is

$$|A - B| = |A| - |B| = 1000 - 27 = 973,$$

by the subtraction principle. $\square$