Math 170S
Lecture Notes Section 8.7 *
Best critical region

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NOTE: Materials that appear in the textbook but do not appear in the lecture notes might still be tested. Please send me an email if you find typos.

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†This notes is based on Hanbaek Lyu’s and Liza Rebrova’s notes from the previous quarter, and I would like to thank them for their generosity. “Nanos gigantum humeris insidentes (I am but a dwarf standing on the shoulders of giants)”.
1 Example: Chocolate, Level 1

Let $X$ be the sugar content of a random chocolate bar, which is a normal random variable with unknown mean $\mu$ and variance 36.

- The company claims that $\mu = 50$ (the null hypothesis $H_0$);

- However, the Federal Trade Commission claims that $\mu = 55$ (the alternative hypothesis $H_1$).

Let $X_1, \ldots, X_n$ be $n = 16$ random samples for $X$. Your friendly instructor wants to find a test so that the type I error $\alpha$ is equal to 0.05.
Suppose that there are two choices of critical regions:

- The critical region $C$,

\[ C = \{(X_1, \ldots, X_n) \mid X_1 + X_2 + \ldots + X_n \geq 839.52\}. \]

This is our usual choice of critical region.

- The critical region $D$,

\[ D = \{(X_1, \ldots, X_n) \mid X_1 + 2X_2 + \ldots + nX_n \geq 7183\}. \]
Both critical regions have type I error $\alpha \approx 0.05$:

- For the critical region $C$,

$$\alpha = P[\text{reject } H_0 \text{ given that } H_0 \text{ is true}] = P[X_1 + X_2 + \ldots + X_n \geq 839.52 \text{ given that } \mu = 50].$$

Note that $X_1 + \ldots + X_n$ is a normal RV with mean $n\mu_0 = (16)(50)$ and variance $n\sigma^2 = (16)(36)$, so

$$\alpha = 1 - \Phi\left(\frac{(839.52) - (16)(50)}{\sqrt{(16)(36)}}\right) = 1 - \Phi(1.647) \approx 0.05.$$
• For the critical region $D$,

$$
\alpha = P[\text{reject } H_0 \text{ given that } H_0 \text{ is true}]
= P \left[ X_1 + 2X_2 + \ldots + nX_n \geq 7183 \text{ given } \mu = 50 \right].
$$

Note $X_1 + 2X_2 + \ldots + nX_n$ is a normal RV with

mean $= E[X_1] + 2E[X_2] + \ldots + nE[X_n]$

$= \frac{n(n+1)}{2} \mu_0 = 6800$;

variance $= \text{var}[X_1] + (2)^2\text{var}[X_2] + \ldots + (n)^2\text{var}[X_n]$

$= \frac{n(n+1)(2n+1)}{6}\sigma^2 = 53856$,

so we have

$$
\alpha = 1 - \Phi \left( \frac{(7183) - (6800)}{\sqrt{53856}} \right)
= 1 - \Phi(1.65) \approx 0.05.
$$
Both $C$ and $D$ have comparable type I error, so we choose the one with smaller the type II error $\beta$.

- For critical region $C$,

$$
\beta = P[\text{not rejecting } H_0 \text{ given that } H_1 \text{ is true}]
= P[X_1 + \ldots + X_n < 839.52 \text{ given that } \mu = 55].
$$

Recall that $X_1 + \ldots + X_n$ is a normal RV with mean $n\mu_1 = (16)(55)$ and variance $n\sigma^2 = (16)(36)$, so

$$
\beta = \Phi\left(\frac{(839.52) - (16)(55)}{\sqrt{(16)(36)}}\right)
= \Phi(-1.69) \approx 0.0455.
$$
For the critical region $D$,

$$\beta = P[\text{not rejecting } H_0 \text{ given that } H_1 \text{ is true}]$$

$$= P [X_1 + 2X_2 + \ldots + nX_n < 7183 \text{ given } \mu = 55].$$

Recall $X_1 + 2X_2 + \ldots + nX_n$ is a normal RV with

$$\text{mean} = \frac{n(n + 1)}{2} \mu_1 = 7480;$$

$$\text{variance} = \frac{n(n + 1)(2n + 1)}{6} \sigma^2 = 53856.$$

$$\beta = \Phi \left( \frac{(7183) - (7480)}{\sqrt{53856}} \right)$$

$$= \Phi(-1.28) \approx 0.1003.$$ 

So the region $C$ has smaller type II error, and hence a better critical region. We will see that $C$ is actually the best critical region for significance level $\alpha = 0.05.$
2 Setting: best critical region

Object: \( X \) is a random variable with density \( f_\theta \) with unknown \( \theta \).

Hypotheses:

• **Null Hypothesis** \( H_0 \): \( \theta \) is equal to \( \theta_0 \).

• **Alternative Hypothesis** \( H_1 \): \( \theta \) is equal to \( \theta_1 \).

Input: Random samples \( X_1, \ldots, X_n \) for \( X \) and significance level \( \alpha \).

Methodology:

Reject \( H_0 \) if \( (X_1, \ldots, X_n) \) is contained in the (to be determined) critical region \( C \), and do not reject \( H_0 \) otherwise.
Problem:

- Find the best critical region $C$ with significance level (type I error) $\alpha$. The test that uses this $C$ is called a uniformly most powerful test.
3 Definition: best critical region

Definition 1. $C$ is the *best critical region of size* $\alpha$ if the type II error is the smallest among all critical regions with significance level $\alpha$. 
4 Theorem: Neyman-Pearson

Recall the definition of the maximum likelihood function

\[ L(\theta) = f_\theta(x_1) \ldots f_\theta(x_n). \]

Remember that \( \theta \) is unknown and \( x_1, \ldots, x_n \) are variables.

**Theorem 2** (Neyman-Pearson lemma). A critical region \( C \) of size \( \alpha \) is the best critical region if there exists \( k \) such that (write this down)

- The type I error \( P[(X_1, \ldots, X_n \in C); \theta = \theta_0] \) is equal to \( \alpha \);
- \( \frac{L(\theta_0)}{L(\theta_1)} \leq k \) for all \( (x_1, \ldots, x_n) \) in \( C \); and
- \( \frac{L(\theta_0)}{L(\theta_1)} \geq k \) for all \( (x_1, \ldots, x_n) \) outside of \( C \).
5 Example: Chocolate, Level 2

Recall the chocolate example, $X$ is a random variable with unknown mean $\mu$ and variance 36. We have $n = 16$ random sample $X_1, \ldots X_n$ for $X$.

- The null hypothesis is $\mu = \mu_0 = 50$;

- The alternative hypothesis $\mu = \mu_1 = 55$.

We will show that the critical region

$$C = \{(X_1, \ldots, X_n) \mid X_1 + X_2 + \ldots + X_n \geq 839.52\}$$

is the best critical region of size 0.05.
We first compute the maximum likelihood function $L(\mu)$. We have (BT)

\[
L(\mu) = f_\mu(x_1) \cdots f_\mu(x_n) \\
= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left( \frac{-(x_i - \mu)^2}{2\sigma^2} \right) \\
= (2\pi\sigma^2)^{-n/2} \exp \left( -\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu)^2 \right) \\
= (72\pi)^{-n/2} \exp \left( -\frac{1}{72} \sum_{i=1}^{n} (x_i - \mu)^2 \right).
\]
The ratio \( \frac{L(\mu_0)}{L(\mu_1)} \) is then equal to (BT)

\[
\frac{L(50)}{L(55)} = \frac{(72\pi)^{-n/2} \exp \left( -\frac{1}{72} \sum_{i=1}^{n} (x_i - 50)^2 \right)}{(72\pi)^{-n/2} \exp \left( -\frac{1}{72} \sum_{i=1}^{n} (x_i - 55)^2 \right)}
\]

\[
= \exp \left( \frac{-1}{72} \sum_{i=1}^{n} [(x_i - 50)^2 - (x_i - 55)^2] \right)
\]

\[
= \exp \left( \frac{-1}{72} \sum_{i=1}^{n} [-2x_i(50 - 55) + (50^2 - 55^2)] \right)
\]

\[
= \exp \left( \frac{-1}{72} \sum_{i=1}^{n} [10x_i - 525] \right)
\]

\[
= \exp \left( \frac{-10(x_1 + \ldots + x_n) + (16)(525)}{72} \right).
\]
We now check the three conditions in Neyman-Pearson lemma:

• For the first condition,

\[
P[(X_1, \ldots, X_n \in C); \mu = \mu_0] = P[X_1 + X_2 + \ldots + X_n \geq 839.52 \text{ given that } \mu = 50]
\]

\[
= 1 - \Phi\left(\frac{(839.52) - n\mu_0}{\sqrt{n\sigma^2}}\right)
\]

\[
= 1 - \Phi\left(\frac{(839.52) - (16)(50)}{\sqrt{(16)(36)}}\right)
\]

\[
= 1 - \Phi(1.647) \approx 0.05.
\]
For the second condition, we choose \( k \) to be

\[
k := \exp\left(\frac{-10(839.52) + (16)(525)}{72}\right).
\]

Now note that \((x_1, \ldots, x_n) \in C\) means that

\[x_1 + \ldots + x_n \geq 839.52.\]

So we have

\[
\frac{L(50)}{L(55)} = \exp\left(\frac{-10(x_1 + \ldots + x_n) + (16)(525)}{72}\right)
\leq \exp\left(\frac{-10(839.52) + (16)(525)}{72}\right)
= k.
\]
• For the third condition,

\((x_1, \ldots, x_n)\) outside of \(C\) means that

\[ x_1 + \ldots + x_n < 839.52. \]

So we have

\[
\frac{L(50)}{L(55)} = \exp \left( -10 \left( x_1 + \ldots + x_n \right) + \frac{(16)(525)}{72} \right) \\
> \exp \left( -10 \left( 839.52 \right) + \frac{(16)(525)}{72} \right) \\
= k.
\]

Hence we conclude that \(C\) is indeed the best critical region of size 0.05.
7 Example: Chocolate, Level 3

Let $X$ be a normal RV with unknown mean $\mu$ and variance 36. We have $n$ random sample $X_1, \ldots X_n$ for $X$.

- The null hypothesis is $\mu = \mu_0$;
- The alternative hypothesis $\mu = \mu_1$.

Assume that $\mu_0 < \mu_1$. Find a best critical region $C$ of size $\alpha$. 
8 Answer: Chocolate, Level 3

Recall the maximum likelihood function:

\[ L(\mu) = (72\pi)^{-n/2} \exp\left(-\frac{1}{72} \sum_{i=1}^{n} (x_i - \mu)^2\right). \]

We can then compute the ratio \( \frac{L(\mu_0)}{L(\mu_1)} \) by (BT)

\[
\frac{L(\mu_0)}{L(\mu_1)} = \frac{(72\pi)^{-n/2} \exp \left(-\frac{1}{72} \sum_{i=1}^{n} (x_i - \mu_0)^2\right)}{(72\pi)^{-n/2} \exp \left(-\frac{1}{72} \sum_{i=1}^{n} (x_i - \mu_1)^2\right)}
= \exp \left(\frac{-1}{72} \sum_{i=1}^{n} \left[-2x_i(\mu_0 - \mu_1) + (\mu_0^2 - \mu_1^2)\right]\right)
= \exp \left(\frac{2(\mu_0 - \mu_1)(\sum_{i=1}^{n} x_i) - n(\mu_0^2 - \mu_1^2)}{72}\right)
= \exp \left(\frac{2n(\mu_0 - \mu_1) \bar{x} - n(\mu_0^2 - \mu_1^2)}{72}\right).
\]
For $C$ to be a best critical region, we need to find $k$ so that

- (2nd cond.) For all $(x_1, \ldots, x_n)$ in $C$, we need (BT)

\[
\exp \left( \frac{2n(\mu_0 - \mu_1)\bar{x} - n(\mu_0^2 - \mu_1^2)}{72} \right) \leq k \leq \frac{L(\mu_0)}{L(\mu_1)} \leq k \leq \log k.
\]

Continuing the calculation,

\[
2n(\mu_0 - \mu_1)\bar{x} \leq 72 \log k + n(\mu_0^2 - \mu_1^2)
\]

\[
\bar{x} \geq \frac{72 \log k + n(\mu_0^2 - \mu_1^2)}{2n(\mu_0 - \mu_1)}
\]

\[
\bar{x} \geq \frac{72 \log k}{2n(\mu_0 - \mu_1)} + \frac{\mu_0 + \mu_1}{2}.
\]
• (3rd cond.) For all \((x_1, \ldots, x_n)\) outside of \(C\), we need (ST)

\[
\frac{L(\mu_0)}{L(\mu_1)} \geq k \\
\exp\left(\frac{2n(\mu_0 - \mu_1) \bar{x} - n(\mu_0^2 + \mu_1^2)}{72}\right) \geq k.
\]

\[
\frac{2n(\mu_0 - \mu_1) \bar{x} - n(\mu_0^2 + \mu_1^2)}{72} \geq \log k \\
2n(\mu_0 - \mu_1) \bar{x} \geq 72 \log k + n(\mu_0^2 + \mu_1^2) \\
\bar{x} \leq \frac{72 \log k + n(\mu_0^2 + \mu_1^2)}{2n(\mu_0 - \mu_1)} \\
\bar{x} \leq \frac{72 \log k}{2n(\mu_0 - \mu_1)} + \frac{\mu_0 + \mu_1}{2}.
\]

Remark 3. A shorter way to do this: “By the same argument as before, we conclude that ...”.
So we conclude

- For \((x_1, \ldots, x_n)\) in \(C\), we need
  \[
  \bar{x} \geq \frac{72 \log k}{2n(\mu_0 - \mu_1)} + \frac{\mu_0 + \mu_1}{2}.
  \]

- For \((x_1, \ldots, x_n)\) outside of \(C\), we need
  \[
  \bar{x} \leq \frac{72 \log k}{2n(\mu_0 - \mu_1)} + \frac{\mu_0 + \mu_1}{2}.
  \]

Thus the best critical region \(C\) is of the form (write this down)

\[
C = \left\{ (X_1, \ldots, X_n) \mid \bar{X} \geq \frac{72 \log k}{2n(\mu_0 - \mu_1)} + \frac{\mu_0 + \mu_1}{2} \right\},
\]

where \(k\) is to be determined.
We can determine $k$ with the 1st cond.,

\[
P[(X_1, \ldots, X_n) \in C; \mu = \mu_0] = \alpha
\]

\[
P \left[ \bar{X} \geq \frac{72 \log k}{2n(\mu_0 - \mu_1)} + \frac{\mu_0 + \mu_1}{2}; \mu = \mu_0 \right] = \alpha.
\]

We write the term in the probability above as

\[
\text{Scary} := \frac{72 \log k}{2n(\mu_0 - \mu_1)} + \frac{\mu_0 + \mu_1}{2}
\]
Since $\overline{X}$ is normal RV with mean $\mu_0$ and variance $\frac{\sigma^2}{n} = \frac{36}{n}$, (write this down)

\[
P[(X_1, \ldots, X_n) \in C; \mu = \mu_0] = \alpha
\]

\[
P[\overline{X} \geq \text{Scary}; \mu = \mu_0] = \alpha
\]

\[
1 - \Phi \left( \frac{\text{Scary} - \mu_0}{6/\sqrt{n}} \right) = \alpha
\]

\[
\frac{\text{Scary} - \mu_0}{6/\sqrt{n}} = \Phi^{-1}(1 - \alpha)
\]

\[
\text{Scary} - \mu_0 = \left( \frac{6}{\sqrt{n}} \right) \Phi^{-1}(1 - \alpha)
\]

\[
\text{Scary} = \mu_0 + \left( \frac{6}{\sqrt{n}} \right) \Phi^{-1}(1 - \alpha).
\]
Substituting the last equation into the formula for critical region, we get

\[ C = \left\{ (X_1, \ldots, X_n) \mid \bar{X} \geq \mu_0 + \left( \frac{6}{\sqrt{n}} \right) \Phi^{-1}(1 - \alpha) \right\}, \]

which is our answer. (Write this down.)
9  Re-example: Chocolate, Level 2

Let $X$ be a normal random variable with unknown mean $\mu$ and variance 36. We have $n = 16$ random sample $X_1, \ldots X_n$ for $X$.

- The null hypothesis is $\mu = 50$;
- The alternative hypothesis $\mu = 55$.

Find a best critical region $C$ of size 0.05.
10 Re-answer: Chocolate, Level 2

Here we have

\[ n = 16; \quad \mu_0 = 50; \quad \alpha = 0.05. \]

So we have

\[ \mu_0 + \left( \frac{6}{\sqrt{n}} \right) \Phi^{-1}(1 - \alpha) = (50) + \left( \frac{6}{\sqrt{16}} \right) \Phi^{-1}(1 - 0.05) \]

\[ \approx 52.47, \]

and therefore the critical region is

\[ C = \{(X_1, \ldots, X_n) \mid \overline{X} \geq 52.47\}, \]

which is equivalent to the critical region from Level 2.
11 Example: Chocolate, Level 4

Let $X$ be a normal random variable with unknown mean $\mu$ and variance 36. We have $n$ random sample $X_1, \ldots, X_n$.

- The null hypothesis is $\mu = 50$;

- The alternative hypothesis $\mu > 50$.

Find a best critical region $\mathcal{C}$ of size 0.05.
The key observation here is that the best critical region $C$ from Level 3

$$C = \left\{ (X_1, \ldots, X_n) \mid \bar{X} \geq \mu_0 + \left( \frac{6}{\sqrt{n}} \right) \Phi^{-1}(1 - \alpha) \right\},$$

does not depend on the mean $\mu_1$ from the alternative hypothesis. We can therefore use $C$ for the best critical region for all $\mu > 50$.

By the same calculation as in Level 2, we conclude that

$$C = \{(X_1, \ldots, X_n) \mid \bar{X} \geq 52.47\}.$$
**Remark 4.** Recall the definition of sufficient statistics $u := u(x_1, \ldots, x_n)$ from Section 6. If a sufficient statistics $u$ exists, then the ratio $\frac{L(\theta_0)}{L(\theta_1)}$ is

$$\frac{L(\theta_0)}{L(\theta_1)} = \frac{\phi(u, \theta_0) h(x_1, \ldots, x_n)}{\phi(u, \theta_1) h(x_1, \ldots, x_n)} = \frac{\phi(u, \theta_0)}{\phi(u, \theta_1)}.$$

By the Neyman-Pearson lemma, this means that the best critical region are usually based on the sufficient statistics when they exist, e.g.,

$$C' = \{(x_1, \ldots, x_n) \mid u(x_1, \ldots, x_n) \geq 42\}, \quad \text{or} \quad C' = \{(x_1, \ldots, x_n) \mid |u(x_1, \ldots, x_n) - 2| \geq 67\}.$$