NOTE: Materials that appear in the textbook but do not appear in the lecture notes might still be tested. Please send me an email if you find typos.

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\(^{\dagger}\)This notes is based on Hanbaek Lyu’s and Liza Rebrova’s notes from the previous quarter, and I would like to thank them for their generosity. “Nanos gigantum humeris insidentes (I am but a dwarf standing on the shoulders of giants)”.

I am but a dwarf standing on the shoulders of giants.”
1 Example: Type I and type II error (recap)

A detective is investigating if the friendly instructor likes cheesy romance movies. Let $X$ be the hatred indicator for a random romance movie.

- $X$ is equal to 0 if the friendly instructor likes the romance movie;
- $X$ is equal to 1 if the friendly instructor hates the romance movie.
The expert team told you that $X$ is Bernoulli RV with parameter $p$

- If the friendly instructor is innocent (the null hypothesis $H_0$), then $p = 1/2$.

- If the friendly instructor is guilty (the alternative hypothesis $H_1$), then $p = 1/4$. 
You then hacked into the friendly instructor’s Netflix account and get the sample values $X_1, \ldots, X_{20}$ for $X$ from 20 romance movies he watched.

- If $\bar{X} \leq 0.3$, then the friendly instructor is guilty and he will be roasted by the whole class for his interest;
- If $\bar{X} > 0.3$, then no actions will be taken against the friendly instructor.

We are interested in two questions:

- What is the probability that you make a false accusation against the friendly instructor? (i.e., reject $H_0$ when $H_0$ is true).
- What is the probability that the friendly instructor gets away scot-free from his crime? (i.e., do not reject $H_0$ when $H_0$ is false).
2 Definition: Type I and Type II error (recap)

Type I error, denoted by $\alpha$, is the probability of rejecting $H_0$ when $H_0$ is true, sometimes also called statistical significance.

Type II error, denoted by $\beta$, is the probability of not rejecting $H_0$ when $H_0$ is false.

It is usually thought that Type I error is more important, as we do not want to falsely accuse people of a crime. However, in this section we will care about both Type I and Type II error.
3 Answer: Type I and type II error (recap)

We first compute $\alpha$. Let $Y$ be the random variable

$$Y := X_1 + X_2 + \ldots + X_{20} = n \overline{X}.$$ 

Note that

- We reject $H_0$ if $Y \leq 20(0.3) = 6$, and we do not reject $H_0$ if $Y > 6$.

- $Y$ is the binomial random variable with parameter $p$ and $n = 20$. 
We can then compute the Type I error $\alpha$ by

$$\alpha = P[\text{Reject } H_0 \mid H_0 \text{ is true}] = P \left[ Y \leq 6 \mid p = \frac{1}{2} \right]$$

$$= \sum_{y=0}^{6} \binom{20}{y} \left( \frac{1}{2} \right)^y \left( \frac{1}{2} \right)^{20-y} = 0.0577.$$

We now compute the type II error $\beta$ by

$$\beta = P[\text{Not rejecting } H_0 \mid H_0 \text{ is false}]$$

$$= 1 - P[\text{rejecting } H_0 \mid H_0 \text{ is false}]$$

$$= 1 - P \left[ Y \leq 6 \mid p = \frac{1}{4} \right]$$

$$= 1 - \sum_{y=0}^{6} \binom{20}{y} \left( \frac{1}{4} \right)^y \left( \frac{3}{4} \right)^{20-y} = 0.2142.$$
4 Example: Type I and Type II error, composite alternative hypothesis

Consider the Netflix example, but with the alternative hypothesis changed,

- The null hypothesis $H_0$: $X$ is a Bernoulli random variable with $p = 1/2$.

- The alternative hypothesis $H_1$: $X$ is a Bernoulli random variable with $p < 1/2$.

Compute the type I error $\alpha$ and type II error $\beta$. 
Answer: Type I and Type II error, composite alternative hypothesis

Type I error $\alpha$ is the same as before.

$$\alpha = P[\text{Reject } H_0 \mid H_0 \text{ is true}] = P\left[Y \leq 6 \mid p = \frac{1}{2}\right]$$

$$= \sum_{y=0}^{6} \binom{20}{y} \left(\frac{1}{2}\right)^y \left(\frac{1}{2}\right)^{20-y} = 0.0577.$$ 

However, $\beta$ now depends on the unknown parameter $p$:

$$\beta = 1 - P[\text{rejecting } H_0 \mid H_0 \text{ is false}]$$

$$= 1 - P\left[Y \leq 6 \text{ given the unknown } p\right]$$

$$= 1 - \sum_{y=0}^{6} \binom{20}{y} (p)^y (1 - p)^{20-y},$$

which is a function of $p$. 
6 Definition: power function

Let $X$ be a random variable with unknown parameter $\mu$.

- Null hypothesis $H_0$: $\mu = \mu_0$;
- Alternative hypothesis $H_1$.

**Definition 1.** The **power function** $K(\mu)$ is

$$K(\mu) := P[\text{rejecting } H_0 \text{ given the unknown } \mu].$$

$K(\mu)$ depends on the rule of rejecting $H_0$. 
Remark 2. Some other textbooks define the power function as

\[ K(\mu) := 1 - P[\text{rejecting } H_0 \text{ given the unknown } \mu]. \]

We will follow the definition in our textbook and use the formula in Definition 1.
7 Theorem: power function

**Theorem 3.** The type I error $\alpha$ and the type II error $\beta$ are given by

$$\alpha = K(\mu_0); \quad \beta = 1 - K(\mu).$$

The power function can be used to both determine $\beta$ and to determine the critical region.
8 Example: power function

Let $X$ be a **chi-square random variable** with unknown $r$ degrees of freedom. Suppose that:

- The null hypothesis $H_0$ is $r = r_0 = 1$;
- The alternative hypothesis $H_1$ is $r > 1$.

Let $X_1, X_2, X_3, X_4$ be 4 random samples for $X$.

1. Compute the critical region for the sample mean $\bar{X}$ with significance level 0.025.

2. Compute the type II error $\beta$ given that $r = 5$. 
9 Answer: chi-square random variable

We are in case (a), so the critical region is of the form

\[ [r_0 + c, \infty) = [1 + c, \infty), \]

for some error \( c \) that we want to compute. We now compute the power function \( K(r) \),

\[
K(r) = P[\text{rejecting } H_0 \text{ given the unknown } r] \\
= P[\overline{X} \geq 1 + c \text{ given } r] \\
= P[X_1 + X_2 + X_3 + X_4 \geq 4(1 + c) \text{ given } r].
\]
Now note that $X_1 + X_2 + X_3 + X_4$ is a chi-square random variable with $4r$ degrees of freedom, so

$$K(r) = P[\chi^2(4r) \geq 4(1 + c)].$$

We can compute $K(r)$ using Table IV in textbook Appendix B. For the type I error $\alpha$,

$$\alpha = K(r_0) = P[\chi^2(4) \geq 4(1 + c)].$$

To get significance level 0.025, we need $\alpha = 0.025$, so

$$P[\chi^2(4) \geq 4(1 + c)] = 0.025.$$

By the power of Table IV, we have $4(1 + c) = 11.14$, so $c = 1.785$, and the critical region is

$$[2.785, \infty).$$
We now compute the type II error $\beta$ when $r = 5$. We have

$$\beta = 1 - K(5) = 1 - P[\chi^2(20) \geq 11.14] \approx 1 - 0.943 = 0.057,$$

as desired.
Let $X$ be a normal random variable with unknown mean $\mu$ and variance 100. Suppose that:

• The null hypothesis $H_0$ is $\mu = \mu_0 = 60$;

• The alternative hypothesis $H_1$ is $\mu > 60$.

Find the number of samples $n$ so that

1. The type I error $\alpha$ is equal to 0.025; and

2. The type II error $\beta$, given that $\mu = 65$, is equal to 0.05.
11 Answer: normal RV

We are in case (a), so the critical region is of the form

\[ [\mu_0 + c, \infty) = [60 + c, \infty), \]

for some error \( c \) that we want to compute. We now compute the power function \( K(\mu) \),

\[ K(\mu) = P[\text{rejecting } H_0 \text{ given the unknown } \mu] \]

\[ = P[\overline{X} \geq 60 + c \text{ given the unknown } \mu]. \]

Since \( \overline{X} \) is a normal RV with mean \( \mu \) and variance \( \frac{100}{n} \), so

\[ K(\mu) = 1 - \Phi \left( \frac{60 + c - \mu}{10/\sqrt{n}} \right), \]

where \( \Phi \) is the cdf of the standard normal RV.
We now compute the type I error by using the power function. We have

\[ \alpha = K(60) = 1 - \Phi \left( \frac{c}{10/\sqrt{n}} \right). \]

We now compute the type II error \( \beta \) when \( \mu = 65 \). We have

\[ \beta = 1 - K(65) = \Phi \left( \frac{c - 5}{10/\sqrt{n}} \right). \]
Plugging $\alpha = 0.025$ and $\beta = 0.05$ from the question,

$$1 - \Phi \left( \frac{c}{10/\sqrt{n}} \right) = 0.025 \quad \text{and} \quad \Phi \left( \frac{c - 5}{10/\sqrt{n}} \right) = 0.05.$$ 

By the power of the magical table, we then have

$$\frac{c}{10/\sqrt{n}} = \Phi^{-1}(1 - 0.025) = 1.96$$

$$\frac{c - 5}{10/\sqrt{n}} = \Phi^{-1}(0.05) = -1.645.$$ 

Solving these inequalities simultaneously, we obtain

$$1.96 - \frac{5}{10/\sqrt{n}} = -1.645$$

$$\frac{10}{\sqrt{n}} = \frac{5}{3.605}$$

$$n \approx 51.98,$$

so we will take $n = 52$ as our answer.
Let $X$ be a **Bernoulli random variable** with unknown parameter $p$.

- The null hypothesis $H_0$ is $p = p_0 = \frac{1}{2}$;
- The alternative hypothesis $H_1$ is $p < \frac{1}{2}$.

We have $n = 31$ samples.

- Reject the null hypothesis $H_0$ if $X_1 + \ldots + X_{31} \leq 11$;
- Test is inconclusive if $X_1 + \ldots + X_{31} > 11$.

Approximate the type II error $\beta$ if $p = \frac{1}{4}$.
13 \textbf{Answer: Bernoulli \textit{RV}}

The power function $K(p)$ is given by

$$K(p) = P[X_1 + \ldots + X_{31} \leq 11 \text{ given unknown } p].$$

Let $Y$ be the random variable $X_1 + \ldots + X_{31}$. We approximate $Y$ by the normal RV with mean and variance

$$E[Y] = np; \quad \text{Var}[Y] = np(1 - p),$$

so we have

$$K(p) = P[Y \leq 11 \text{ given unknown } p]$$

$$= \Phi \left( \frac{11 + 0.5 - np}{\sqrt{np(1 - p)}} \right).$$
Plugging $n = 31$ and $p = \frac{1}{4}$, we have

$$K\left(\frac{1}{4}\right) = \Phi \left( \frac{11 + 0.5 - (31)\left(\frac{1}{4}\right)}{\sqrt{(31)\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)}} \right) = \Phi(1.28) = 0.8997,$$

so the Type II error is

$$\beta = 1 - K\left(\frac{1}{4}\right) = 0.1003,$$

as desired.