PHUTBALL DRAWS

SUCHARIT SARKAR

ABSTRACT. In this short note, we exhibit a draw in the game of Philosopher's Phutball. We construct a position on a 12×10 Phutball board from where either player has a drawing strategy, and then generalize it to an $m \times n$ board with $m - 2 \geq n \geq 10$.

Philosophers' Phutball, invented in the towers of Cambridge by Conway and company, and named after a much-beloved Monty Python sketch of a similar name, is a two-player game played on a $m \times n$ board. The official Phutball pitch, as described by Berlekamp-Conway-Guy [BCG03], is 19×15 (that is, there are 19 rows and 15 columns); however, Phutball is usually played on a 19×19 Go board.

The rules of the game are fairly simple. One of the grid points is occupied by *the ball*, usually a black Go stone. Alfred, the first player, wins if the ball crosses the topmost row or ends up in the topmost row at the end of a turn. Betty, going second, wins if the ball crosses the bottommost row or ends up in the bottommost row at the end of a turn. On their turn, the players may either place *a chap*, usually a white Go stone, or move the ball. The ball is moved by jumping over a line of chaps in one of the eight possible directions, and those chaps are immediately removed; and multiple jumps are allowed, although not required. However, in all our diagrams, we will represent the ball by a grey stone and the chaps by black stones (Phutball played with reversed colors is called floodlit Phutball).

Despite the simplicity of the rules, the game is fairly complicated. Phutball on an $n \times n$ board is PSPACE-hard [Der10] and to even check if one has a win in one is NP-complete [DDE02]. Several variants of Phutball have been analyzed in great detail, such as "directional Phutball" [Loo08] when the players are constrained to jump in certain directions, or "one-dimensional Phutball" [GN02] played on an $m \times 1$ board. Even the one-dimensional game is surprisingly complicated: It has only been analyzed fully in [GN02] when the players are forbidden to place 'off-parity' chaps. In the general one-dimensional game, sometimes the only winning move for a player is to jump all the way back! This makes the game 'loopy' and hard to analyze. It is not even known if there are one-dimensional configurations from where both players have a drawing strategy [Sie09]. Even less is known regarding the usual two-dimensional Phutball. Starting from the middle on a one-dimensional empty $(2m+1) \times 1$ board, the first player Alfred can at least ensure a draw by the usual strategy-stealing argument, similar to the proof of [Loo08, Observation 3], but even the strategy-stealing argument fails in the two-dimensional case. After filling up his top row with chaps, Alfred runs out of 'passes'. So it is still unknown if starting from the middle on a two-dimensional empty $(2m+1) \times n$ board the first player Alfred has a drawing strategy.

In this short note, we will study draws in the game of Phutball. The main result, Theorem 1, is that there is a position on a 12×10 board which leads to a draw under optimal play by both the players. The configuration immediately generalizes to an $m \times n$ board with $m - 2 \geq n \geq 10$, see Corollary 2.

We will use chessboard notation to describe the board. The columns are labeled A, B, C, ..., left to right, the rows are numbered 1, 2, 3, ..., bottom to top, and the grid points are labeled a1, a2, ..., b1, b2, ..., accordingly. The kth move by Alfred will be referred to as $\alpha(k)$, and the kth move by Betty as $\beta(k)$, with the k in Roman numerals. A chap placement will simply be referred to by the name of the grid point where the chap was placed, while a jump will be described by a (non-empty) sequence of arrows, where the arrows describe the directions of the jump (read left to right).

Allow us to illustrate the rules, the notations, and a few subtleties in the following example. Assume we have the board position from Figure 1 on a 5×5 board. At this point, these are the legal moves: a chap placement on any of the 13 empty points, or any of the following jumps, $\downarrow$, $\searrow$, $\searrow\uparrow$, $\searrow\uparrow\searrow$, $\searrow\uparrow\uparrow$, $\searrow\uparrow\nearrow$. The jumps $\searrow\uparrow\uparrow$ and $\searrow\uparrow\nearrow$ win the game for Alfred, while the jumps $\downarrow$, $\searrow$, and $\searrow\uparrow\searrow$ win it for Betty. Note that during the jumps $\searrow\uparrow$, $\searrow\uparrow\uparrow$, and $\searrow\uparrow\nearrow$, the ball uses the bottom row, but since it does not end up in the bottom row at the end of the turn, it is not a win for Betty. Also note that the jumps $\searrow\nearrow$, $\searrow\uparrow\uparrow\leftarrow$, and $\searrow\uparrow\uparrow\searrow$ are not allowed since they hit the sidelines, but the jump $\searrow\uparrow\nearrow$ is allowed; the corners
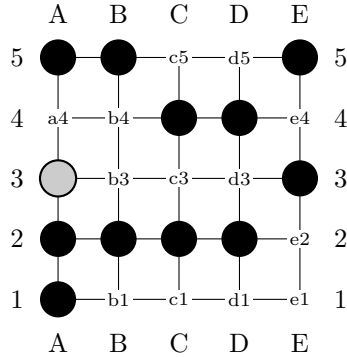
FIGURE 1. An illustrative Phutball position.

are considered part of the goallines, not the sidelines, and jumping diagonally through the corners is allowed; jumping through the top corners is a win for Alfred while jumping through the bottom corners is a win for Betty. The jump $\searrow\uparrow\swarrow$ is also not allowed, since the chap at b2 is removed immediately after the first jump and cannot be reused during the third jump.

Let us now discuss an elementary Phutball strategy before we proceed. If a player has a winning jump, then it is called *a shot*. There are only two ways to defend against a shot. One may perform a jump of their own, jumping away from the danger; we will call such a jump *a jot*, short for *j*umping *o*ut of *t*rouble. Alternatively, one may place a chap along the route of the winning jump, so that the winning jump no longer exists; such a chap placement is called *a tackle*. If after every jump (respectively, a chap placement), it is still a shot, perhaps by some other route, then the original shot is called an *unjottable* (respectively, *untackleable*) shot. We will annotate shots, unjottable shots, and untackleable shots by !, $*$!, and !!, respectively. If a player has a shot that is both unjottable and untackleable, then he or she has a win in one, and we will indicate this by #. Consider the situation from Figure 2 on the 5×5 board. It is Alfred's turn, and he has
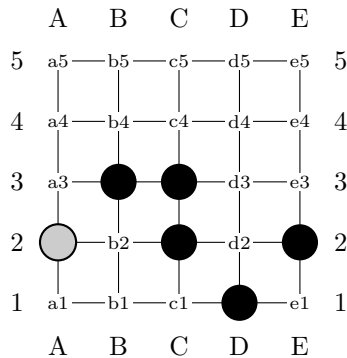

FIGURE 2. Shots, tackles, and jots.

to defend against Betty's shot $\nearrow\downarrow$. He can jot $\nearrow\downarrow\rightarrow\uparrow$, or he can tackle by placing a chap at c4. In this case, the tackle wins while the jot loses. By tackling at c4, Alfred gets the untackleable unjottable shot $\nearrow$. On the other hand, if Alfred had jotted off to e3, then Betty can simply place a chap on e2 to get a untackleable unjottable shot of her own.

Now we are ready for our main result.

Theorem 1. *There are drawn configurations in Phutball played on a 12×10 board.*

Proof. Consider the configuration from Figure 3, with Alfred to play. (Notice that the configuration of the chaps has a symmetry, which will of relevance very soon.)

For her next move, Betty is threatening to place a chap at a1 to get an untackleable shot. Alfred must jot away from that threat, so he needs to place a chap next to the ball to create a jump of his own. Placing
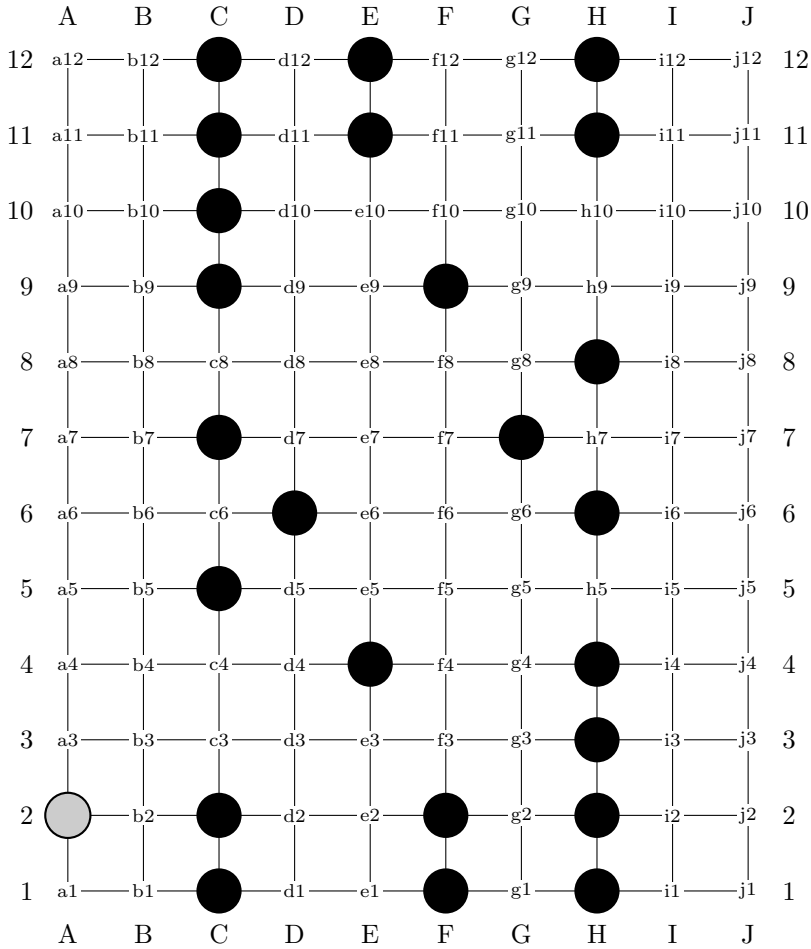
FIGURE 3. A drawn position in Phutball.

a chap at a1, b1, or b2 is a shot for Betty, while placing a chap at a3 is not much better, since the sequence α(i)a3 β(i)a1!! α(ii)↑ β(ii)a3# loses for Alfred. So he must place a chap at b3.

However, this creates a shot for Alfred, so Betty has to defend against it. Jotting off ↗↑→ to e6 is not helpful since Alfred can place a chap back at d6 to get an untackleable unjottable shot. So Betty has to tackle by placing a chap at c4.

The tables turn again. Now Betty has a shot and Alfred has to defend against it. His only available tackle is at d5, and as the following game tree shows, if he does not tackle, he loses.

α(i)b3! β(i)c4! α(ii)d5!

β(ii)e6! α(iii)f7! β(iii)g8!

α(iv)h9*! β(iv)i10 α(v)$\nearrow$
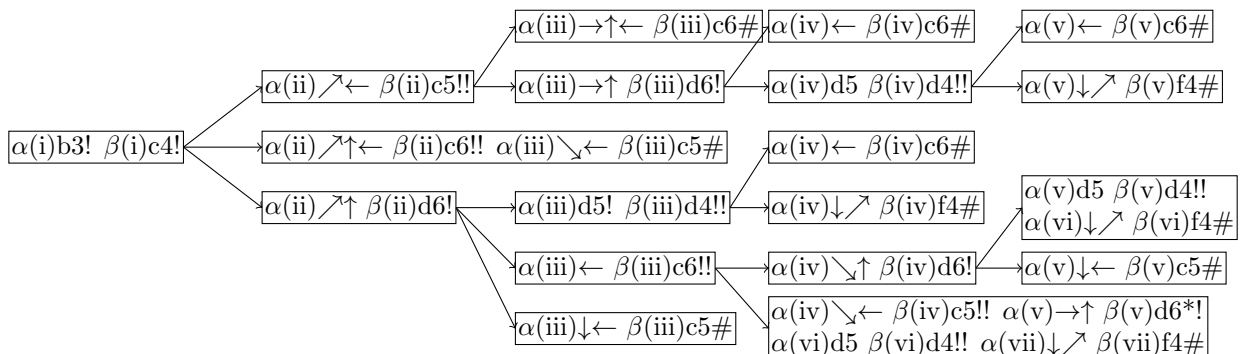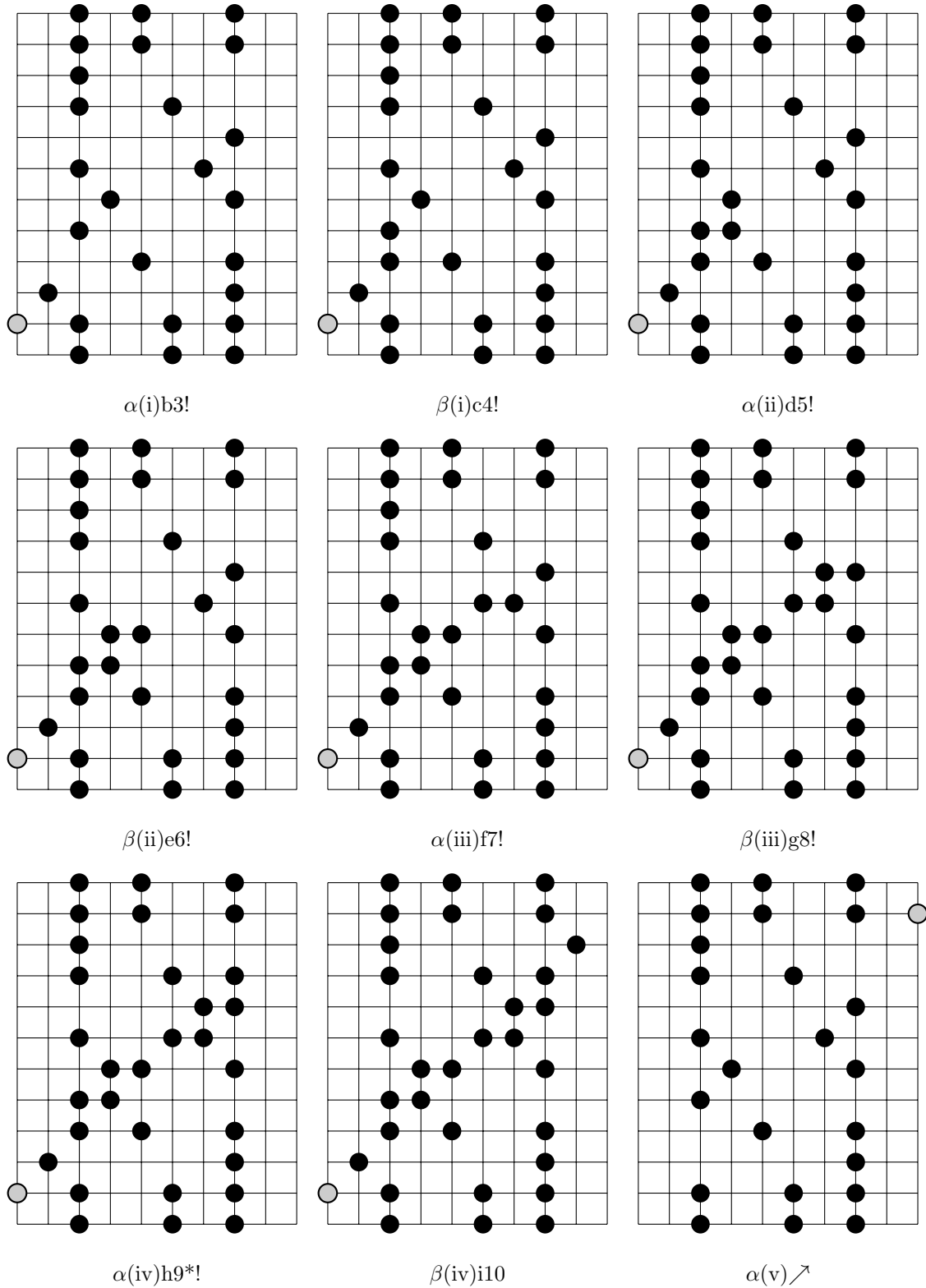
FIGURE 4. Half of the forced sequence of moves in the draw.

That is, whenever Alfred jumps to b5, b7, d7, or f5, Betty places a chap at c5, c6, d6, or f4, respectively; and if the ball is at d7 and there is a chap at d6 and Alfred tackles at d5, then Betty responds by re-tackling at d4.

Therefore, Alfred is forced to tackle at d5, but once he does, he has a shot of his own. Betty can jot off $\nearrow\leftarrow\downarrow$ to c4, but Alfred can simply respond by placing a chap at c5 to get an untackleable unjottable shot. So Betty is also forced to tackle at e6.

And the tables keep on turning. Now Betty gets a shot and jotting off to h9 does not help, so Alfred tackles at f7. Then he gets a shot, and by the previous analysis and the symmetry of the board, we know that none of Betty's three possible jots to g6, i6, or i8 help. Therefore, Betty tackles at g8 and gets another shot. Alfred's jot to f7 is no good, so he tackles at h9, gaining an unjottable shot. Betty places a chap at i10 which is her only defence.

Alfred now has to be extremely careful about his next move. Betty is threatening a win in two by first placing a chap at j11, which blocks all jumps along the $\nearrow$ direction, and then placing a chap at a1 for a shot. The only way Alfred can prevent Betty from placing a chap at j11 is by jumping there himself. If he does not do the jump, then he has two moves to create a new jump for himself against Betty's threat.

In one of those two moves, Alfred has to place a chap next to the ball. Placing a chap at a1 and b1 are always shots for Betty, so he has to place it at b2 or a3. Placing a chap at b2 is also a shot for Betty with the jump $\rightarrow\swarrow$; if Alfred tries to prevent that by first placing a chap at d2, and then at b2, it still remains a shot for Betty via the jump $\rightarrow\searrow$.

Placing a chap at a3 is a shot for Betty as well using the jump $\uparrow\searrow$; Alfred can try to defend against that by first placing a chap at a4, and then at a3. So this is Alfred's only possible defence against Betty's threat, if he chooses not to jump to j11 immediately.

However, the moment Alfred places a chap at a4, Betty places a chap at a1, forcing Alfred to jump $\nearrow$ to j11. Then we get a board position which is almost symmetric to the original position, except we have two additional chaps at a1 and a4. By the previous analysis, Betty and Alfred are forced to place chaps, one at a time, from i10 to b3. The extra chap at a4 does not feature in any of relevant sequences, but the extra chap at a1 comes back to haunt Alfred at the very end, for when he tries to place a chap at b3, the extra chap at a1 makes it a shot for Betty. That is, we get the following sequence

$$\alpha(\text{v})\text{a4}\ \beta(\text{v})\text{a1!!}\ \alpha(\text{vi})\nearrow\ \beta(\text{vi})\text{i10!}\ \alpha(\text{vii})\text{h9!}\ \beta(\text{vii})\text{g8!}\ \alpha(\text{viii})\text{f7!}\ \beta(\text{viii})\text{e6!}\ \alpha(\text{ix})\text{d5!}\ \beta(\text{ix})\text{c4\#}$$

which loses for Alfred.

Therefore, Alfred is forced to play $\alpha(\text{v})\nearrow$. It is now Betty's move, and the position is symmetric to the starting position. Consequently, this is a draw. (The optimal sequence of moves is shown in Figure 4.)   □

Corollary 2. *There are drawn configurations in Phutball played on an $m \times n$ board with $m - 2 \geq n \geq 10$.*

Proof. Let us just show how to generalize the configuration from Figure 3 to the configuration of Figure 5 on the official 19×15 board, and leave the rest as an exercise to the reader. The reader should note that despite filling up the rightmost column with chaps, the sideline subtleties from Figure 1 are not invoked. The reader should additionally note that some of the analysis from the proof of Theorem 1 become more involved. For instance, after $\alpha(\text{i})\text{b3!}$ $\beta(\text{i})\text{c4!}$, if Alfred does not tackle at d5, he still loses, but the following sequence is longer.

$$\alpha(\text{ii})\nearrow\uparrow\ \beta(\text{ii})\text{d6!}\ \alpha(\text{iii})\text{d5!}\ \beta(\text{iii})\text{d4!!}\ \alpha(\text{iv})\downarrow\nearrow\ \beta(\text{iv})\text{f4!!}\ \alpha(\text{v})\nearrow\ \beta(\text{v})\text{h6!!}\ \alpha(\text{vi})\nearrow\ \beta(\text{iv})\text{j8\#}$$ □

The careful reader will notice that the above configurations do not generalize to the usual 19×19 board since the optimal sequence of chap placements in the draw is done along a diagonal, cf. Figure 4, and the diagonal needs go from one sideline to the other since Alfred is only forced to jump because of the sidelines, and we need to have a row below and a row above to ensure that the game is not already over. So we leave that as a question to the reader.

Question 3. *Are there drawn configurations in Phutball played on a 19×19 board?*

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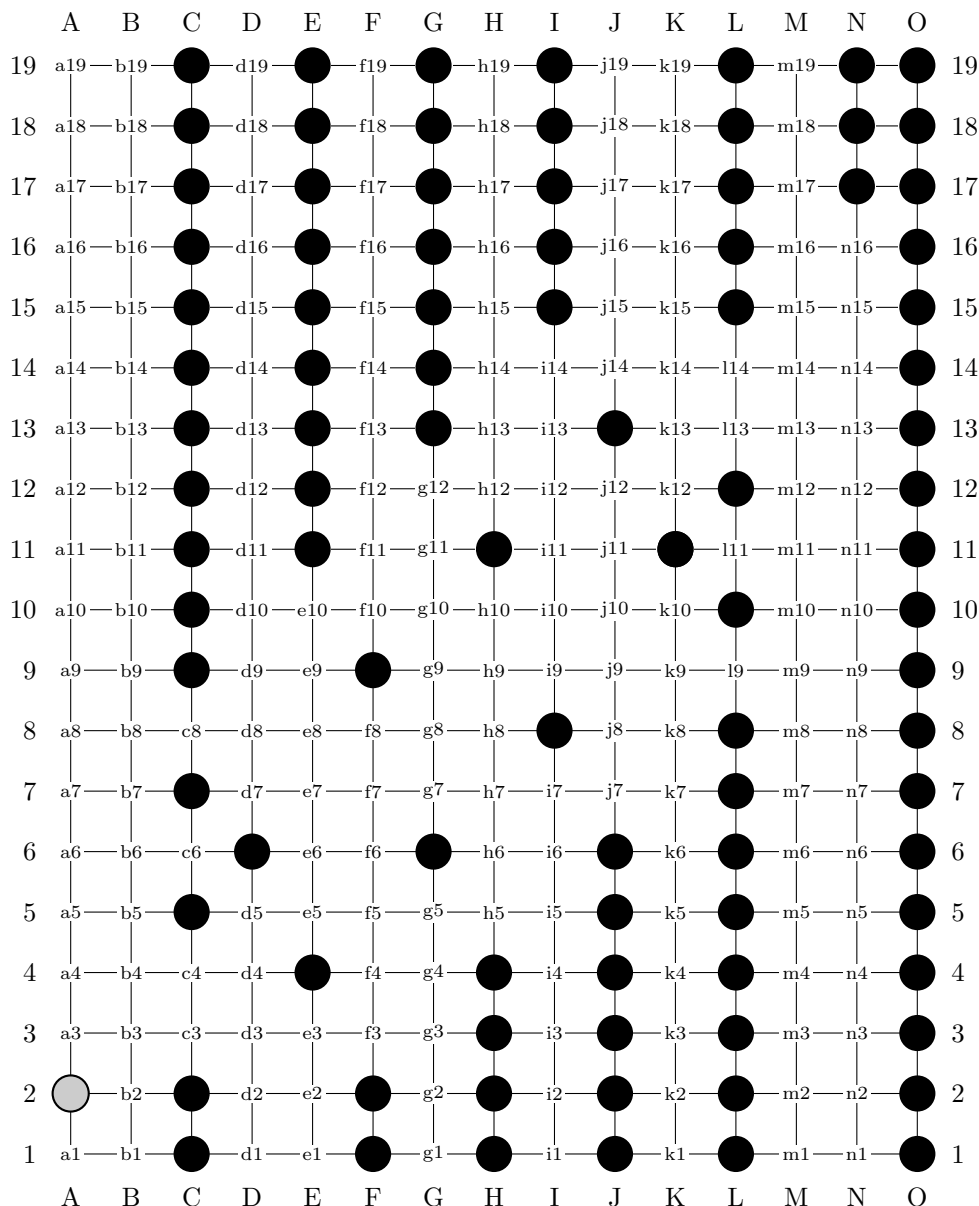
FIGURE 5. A drawn position in Phutball on the standard pitch.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, LOS ANGELES, CA 90095
E-mail address: sucharit@math.ucla.edu