Stable Homotopy Refinements and Khovanov homology

Robert Lipshitz[1] and Sucharit Sarkar[2]

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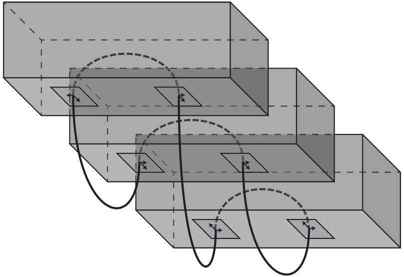
Special thanks to our collaborator Tyler Lawson, whose perspective is reflected throughout.

[1] RL was supported by NSF CAREER Grant DMS-1642067 and NSF FRG Grant DMS-1560783

[2] SS was supported by NSF CAREER Grant DMS-1643401 and NSF FRG Grant DMS-1563615

Part 1: Stable homotopy refinements



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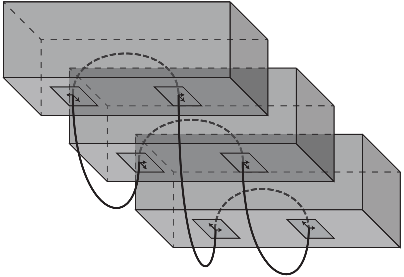
- Morse homology



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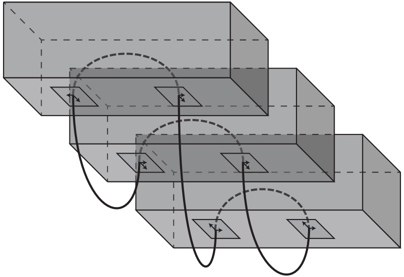

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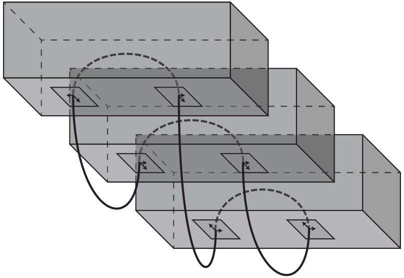

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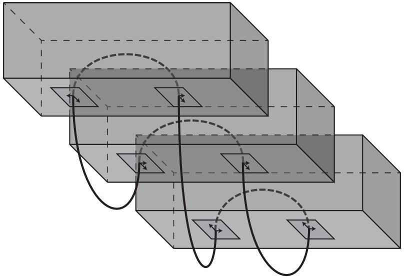

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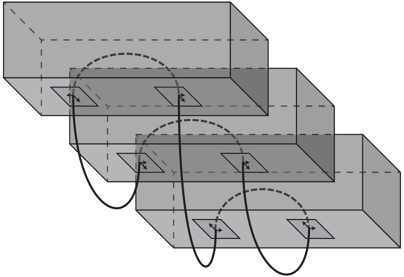

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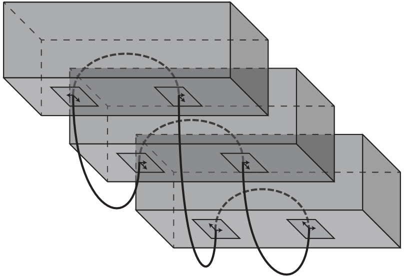

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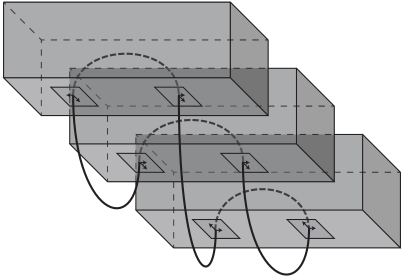

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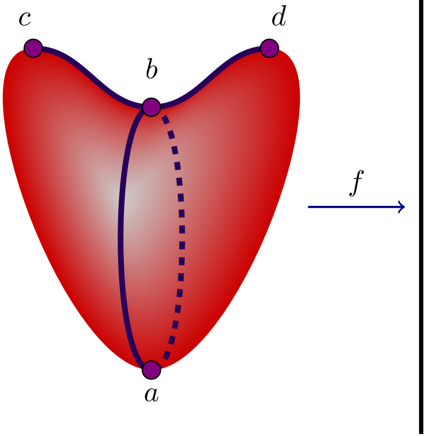

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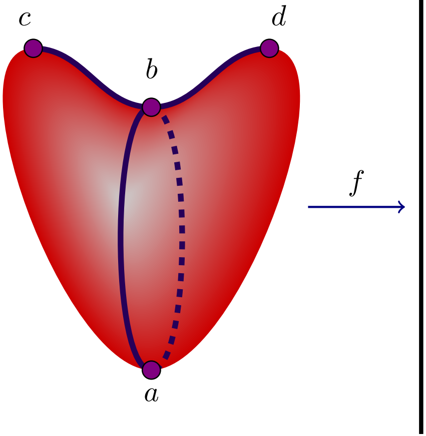

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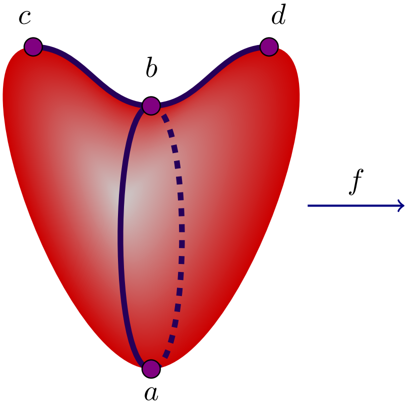

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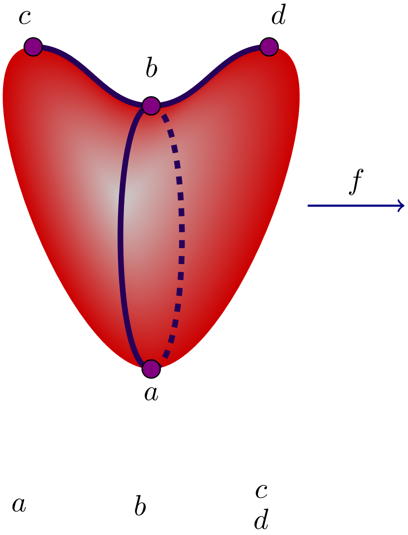

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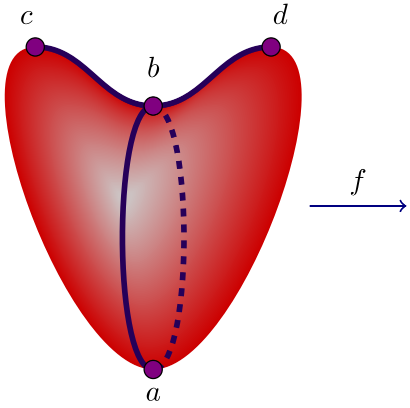

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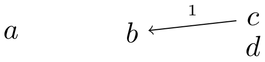

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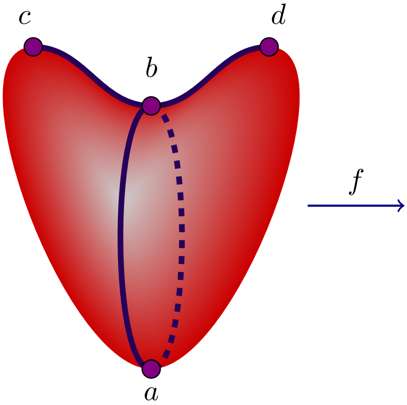

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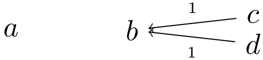

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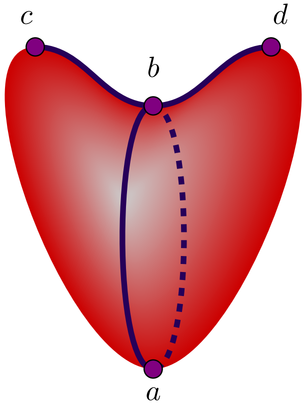

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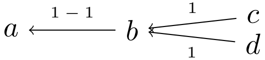

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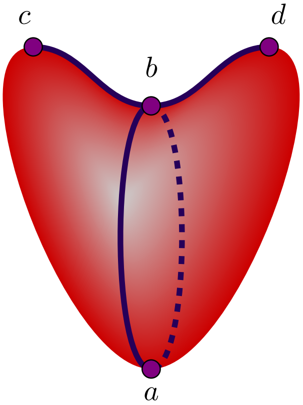

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$$a \xleftarrow{1-1} b \; \begin{smallmatrix} \xleftarrow{1} c \\ \xleftarrow{1} d \end{smallmatrix}$$

Homology

$\mathbb{Z}$ $\quad 0 \quad$ $\mathbb{Z}$

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	Seidel-Smith '06	Symplectic Khovanov homology	$\xrightarrow{\chi}$	Knot determinant

(and many others. . .)

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Question. (Cohen-Jones-Segal) Are these Floer homologies the homologies of naturally associated spaces?
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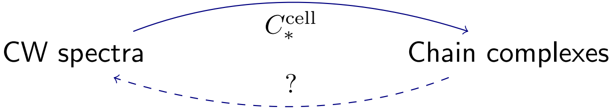
Spatial Refinement Problem. Given a chain complex C_* with distinguished basis, arising in an interesting way, construct a CW spectrum X with $C_*^{\mathrm{cell}}(X) \cong C_*$ with the distinguished basis given by the cells.

A theorem of Carlsson's

Question. Is there a universal way of refining chain complexes, i.e.,



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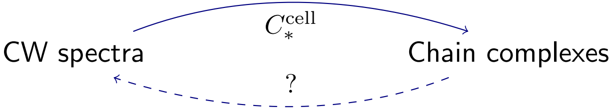

Theorem. No.

Proof.

• (Carlsson '81) Let $G = \mathbb{Z}/2 \times \mathbb{Z}/2$. There is a $\mathbb{Z}[G]$-module P which is not the homology of any G-equivariant (Moore) space.

• P is the homology of a chain complex over $\mathbb{Z}[G]$.

Applications of spatial refinements

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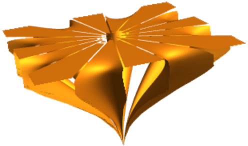
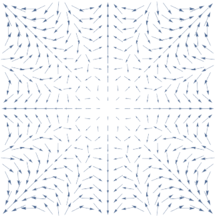
- For group actions on spaces, there are meaningful notions of fixed sets, and localization theorems on equivariant cohomology (Smith theory).

General strategies for spatial refinements





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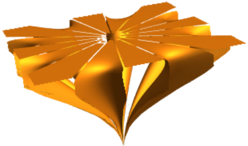

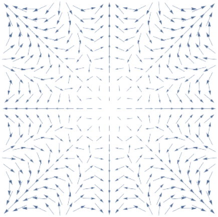

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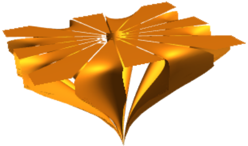

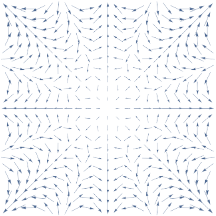

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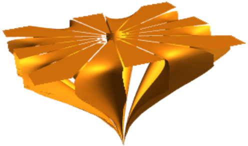

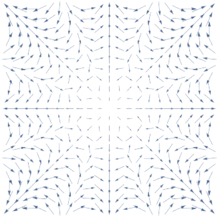

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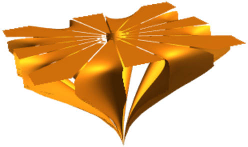

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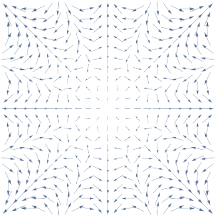

- Hu-Kriz-Kriz '16, Lawson-Lipshitz-Sarkar used functors from the Burnside category to spaces to refine Khovanov homology. One could try to factor through other categories, as well.

Flow categories and their realizations

A *framed flow category* is a way of encoding the moduli space of flows in Morse theory or Floer theory. Cohen-Jones-Segal turn a framed flow category into a CW spectrum.

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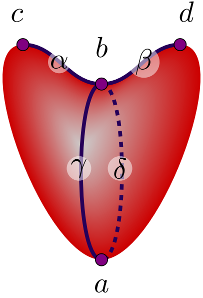

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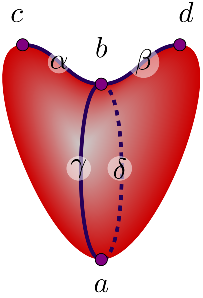

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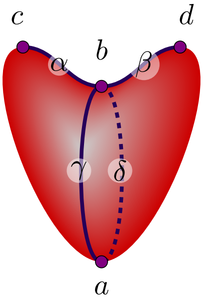

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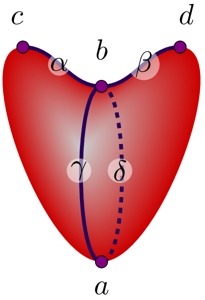

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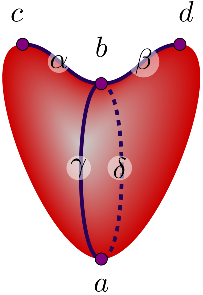

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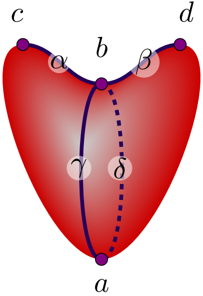

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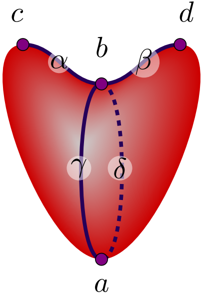

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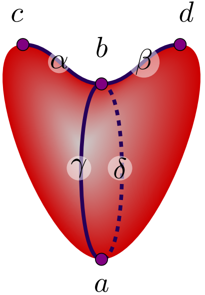

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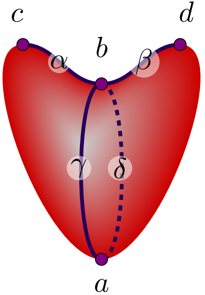

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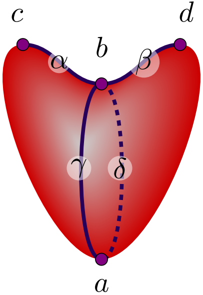

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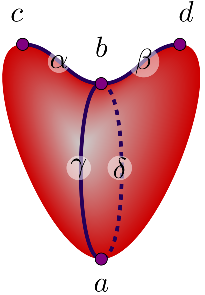

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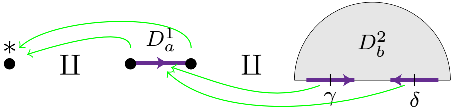

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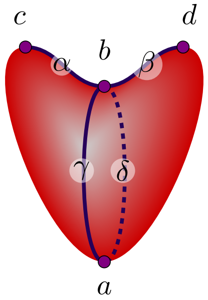

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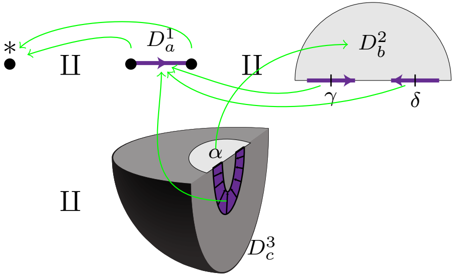

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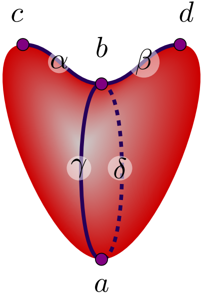

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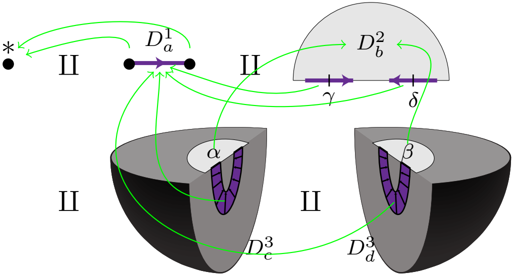

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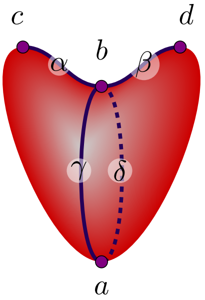

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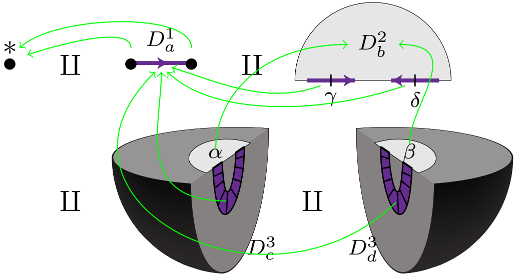

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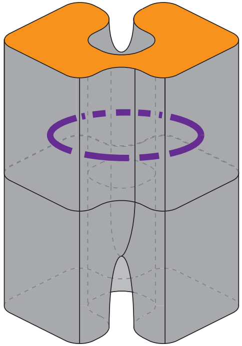
Part 2: Khovanov homology and homotopy



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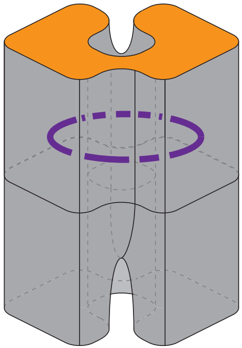

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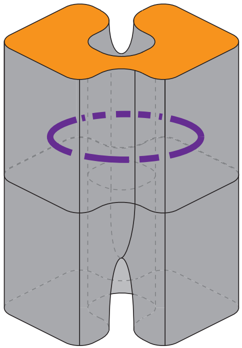

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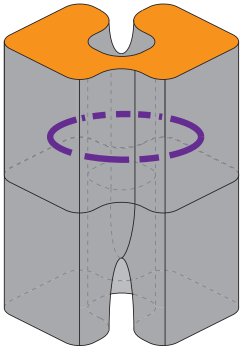

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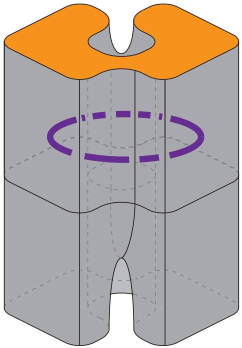

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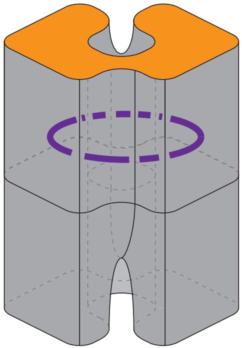

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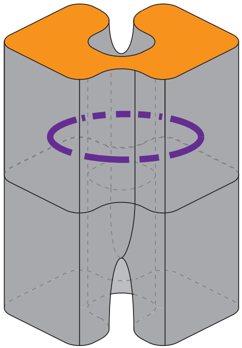

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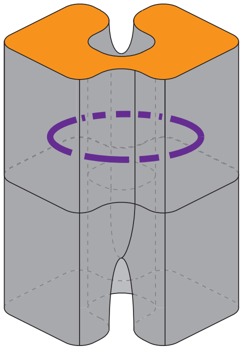

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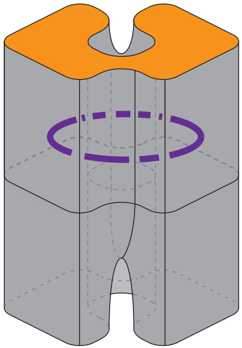

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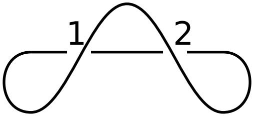

Knot K

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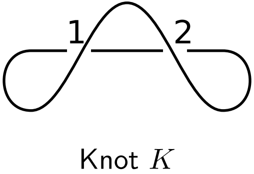

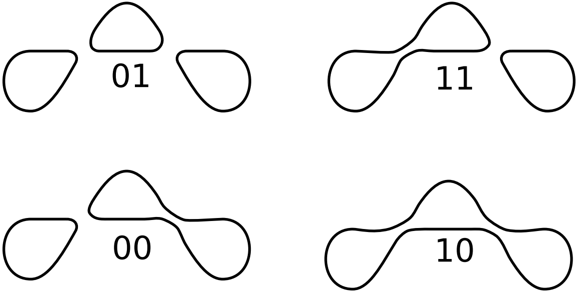

Cube of resolutions (Kauffman '87)

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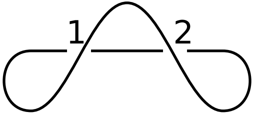

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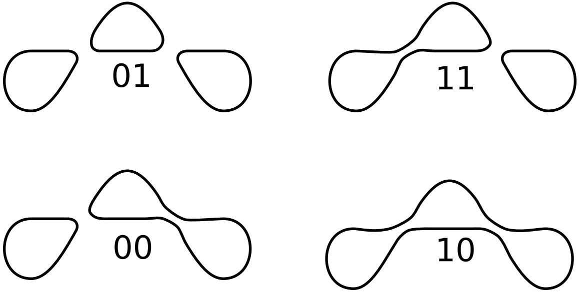

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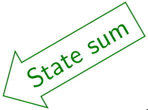

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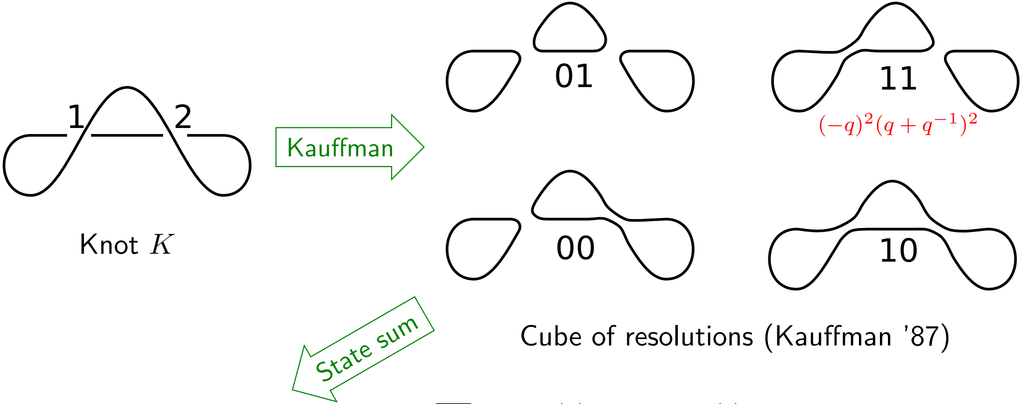

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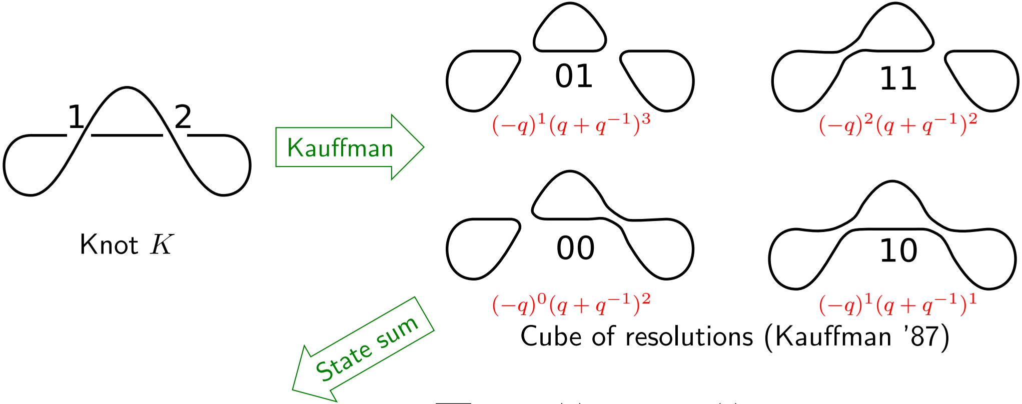

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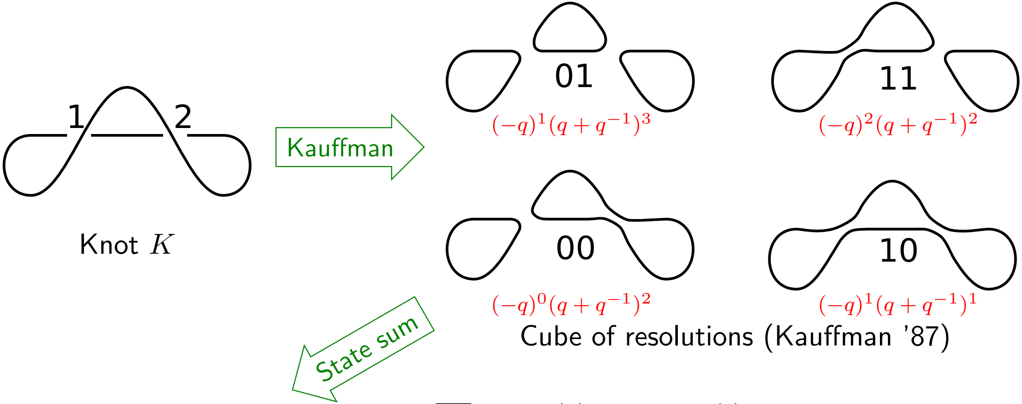

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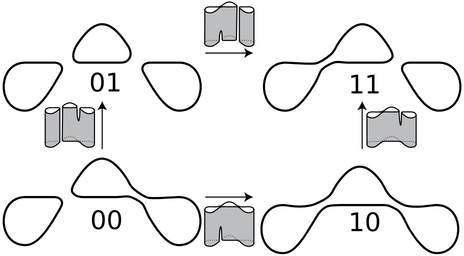

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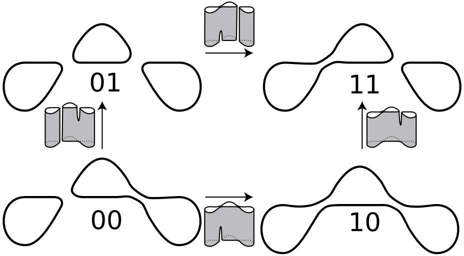

$$\begin{array}{ccc} V \otimes V \otimes V & \xrightarrow{m \otimes \mathrm{Id}} & V \otimes V \\ \uparrow{\scriptstyle \mathrm{Id} \otimes \Delta} & & \uparrow{\scriptstyle \Delta} \\ V \otimes V & \xrightarrow[m]{} & V \end{array}$$

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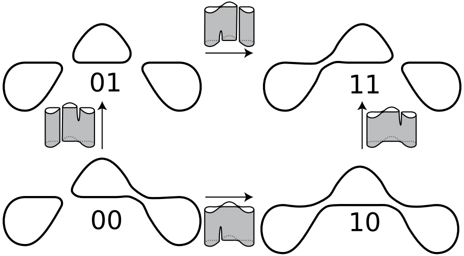

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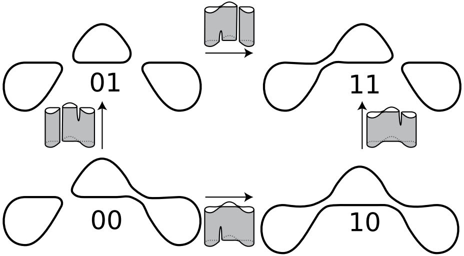

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Famous applications of Khovanov homology

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Theorem. (Rasmussen '04) If K is a positive knot, then

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• **Example.** $g_4(T_{p,q}) = g_3(T_{p,q}) = u(T_{p,q}) = \frac{(p-1)(q-1)}{2}$.
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Theorem. (Kronheimer-Mrowka '10) If $\mathrm{rank}(Kh(K)) = 2$, then K is the unknot.
• Proof uses instanton gauge theory.
• **Old conjecture.** If $V_K(q) = q + q^{-1}$, then K is the unknot.

$u(T_{5,68}) = 134.$

Formal structure of Khovanov homotopy type

Link diagram L ⇒ Finite CW spectrum $X^j_{Kh}(L), j \in \mathbb{Z}$ ⇒ Khovanov homology $Kh^{i,j}(L)$

Formal structure of Khovanov homotopy type

Link diagram L	$\Rightarrow$	Finite CW spectrum $\mathcal{X}^j_{Kh}(L), j \in \mathbb{Z}$	$\Rightarrow$	Khovanov homology $Kh^{i,j}(L)$
Reidemeister moves $L \to L'$	$\Rightarrow$	Weak equivalences $\mathcal{X}^j_{Kh}(L) \simeq \mathcal{X}^j_{Kh}(L')$	$\Rightarrow$	Khovanov isomorphism $Kh^{i,j}(L) \cong Kh^{i,j}(L')$

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Reidemeister moves $L \to L'$	$\Rightarrow$	Weak equivalences $X^j_{Kh}(L) \simeq X^j_{Kh}(L')$	$\Rightarrow$	Khovanov isomorphism $Kh^{i,j}(L) \cong Kh^{i,j}(L')$
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$$X \leftarrow A \rightarrow Y$$

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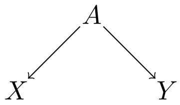

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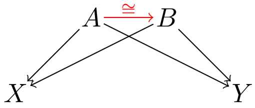

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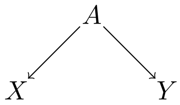

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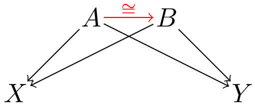

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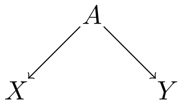

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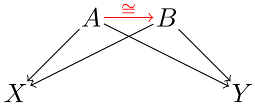

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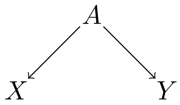

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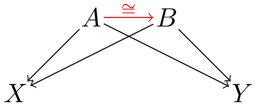

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Elmendorf-Mandell K-theory

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- Can describe the map $\mathcal{B} \to \mathcal{S}pectra$ more explicitly, via Pontryagin-Thom construction (cf. Lawson-Lipshitz-Sarkar).

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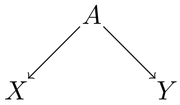

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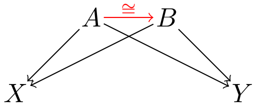

Cube shaped diagrams

$$\begin{array}{ccc} C_{01} & \xrightarrow{f_{\bullet 1}} & C_{11} \\ {\scriptstyle f_{0\bullet}}\uparrow & & \uparrow{\scriptstyle f_{1\bullet}} \\ C_{00} & \xrightarrow[f_{\bullet 0}]{} & C_{10} \end{array}$$

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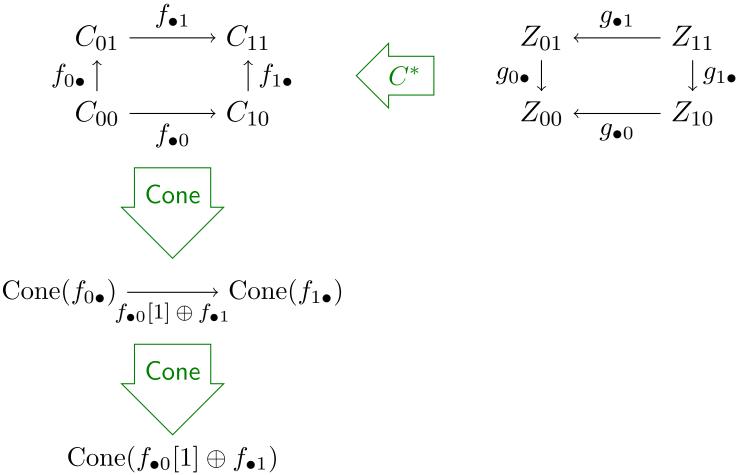

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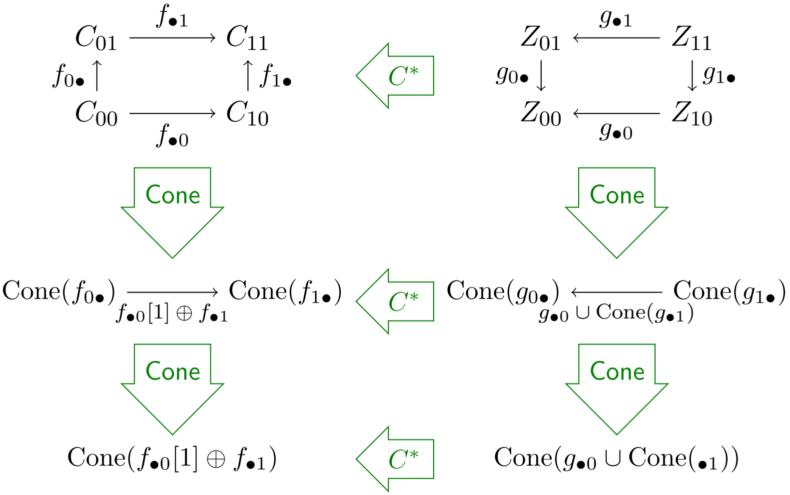

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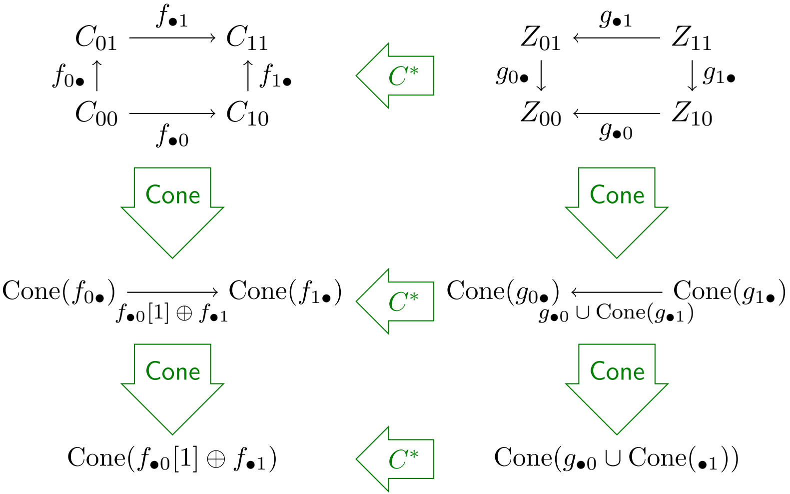

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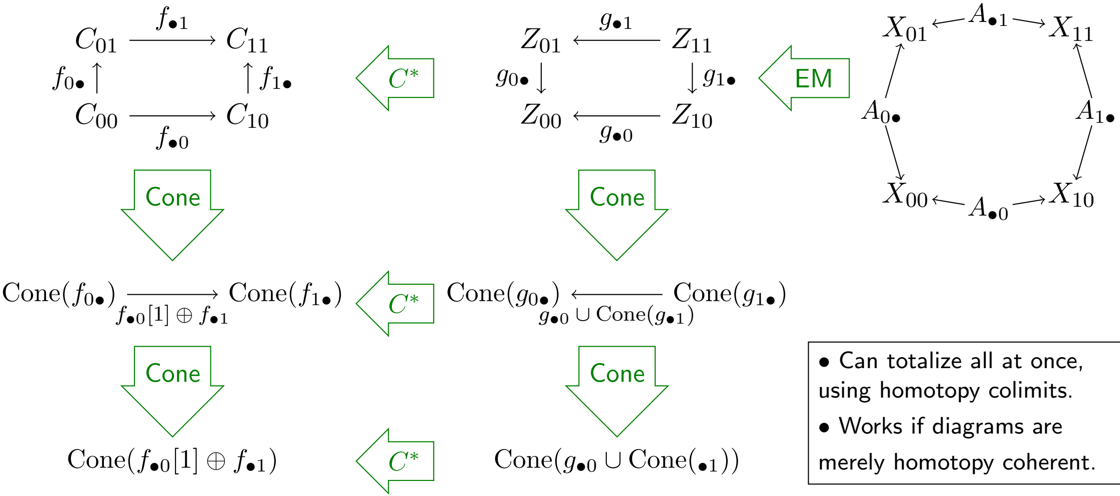

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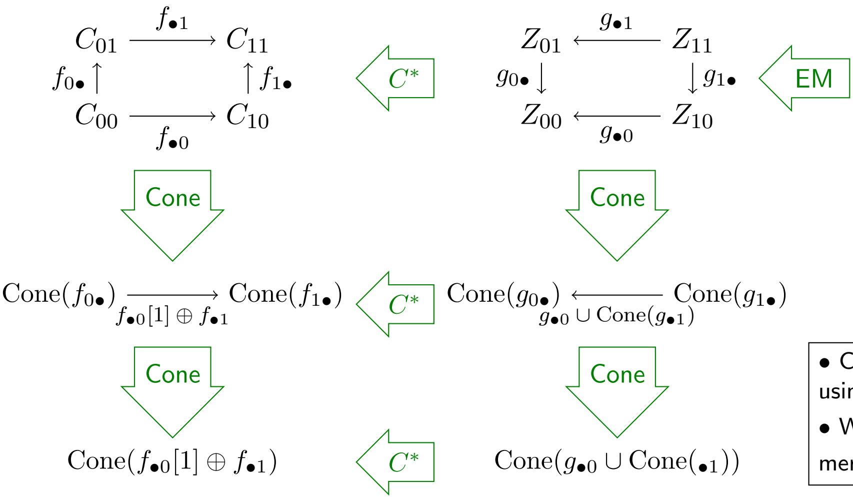

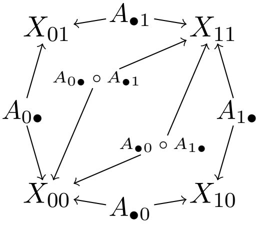

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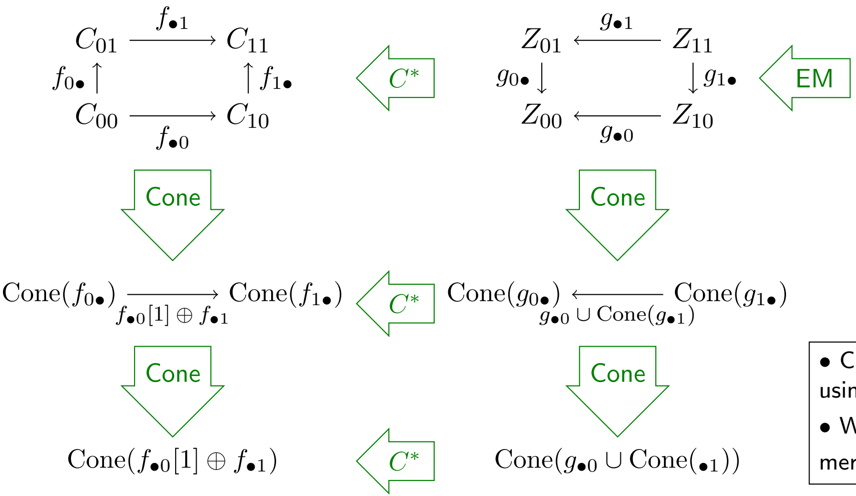

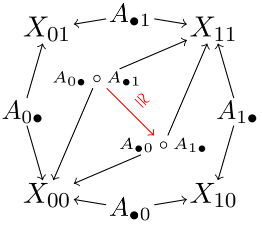

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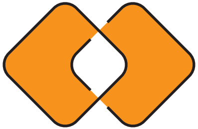

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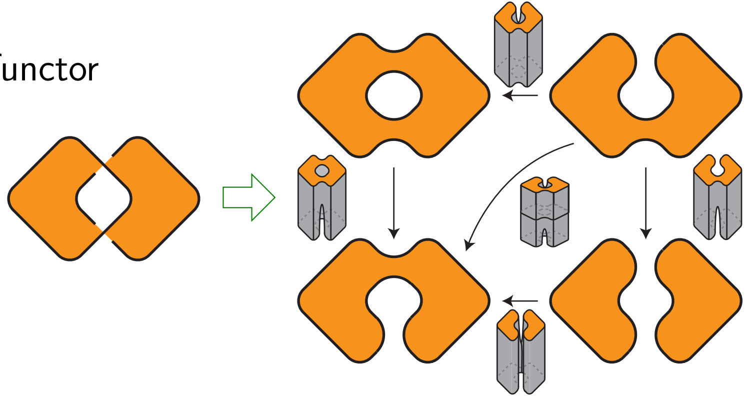

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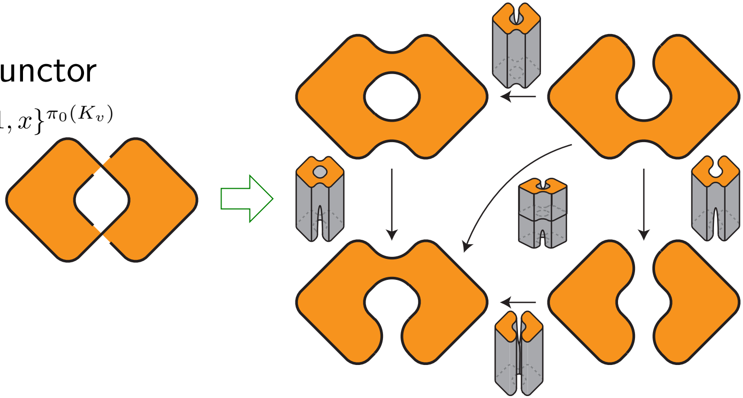

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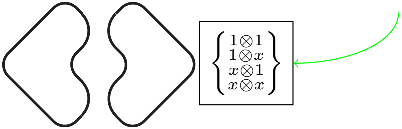

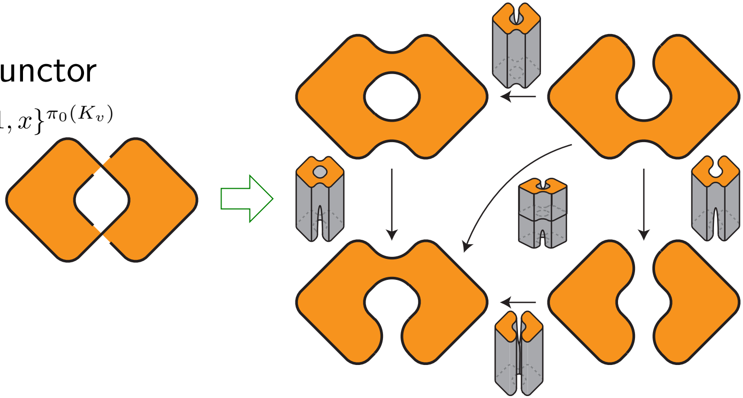

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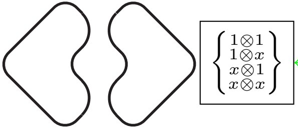

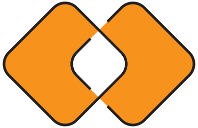

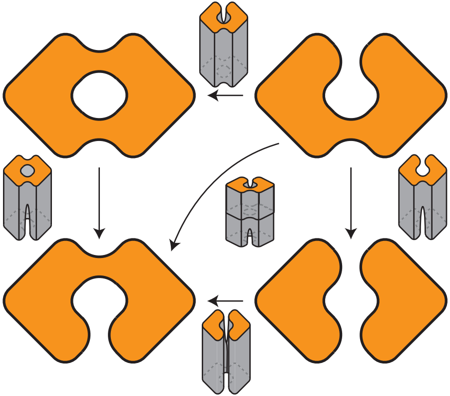

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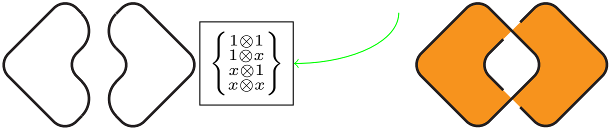

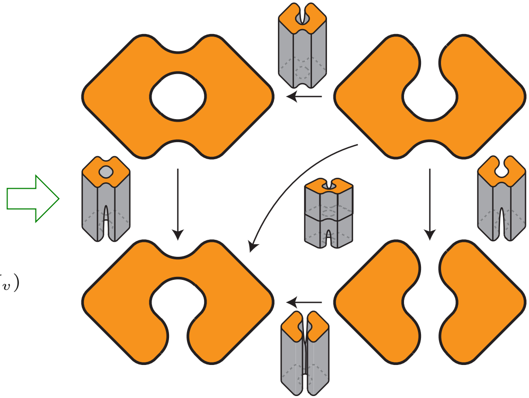

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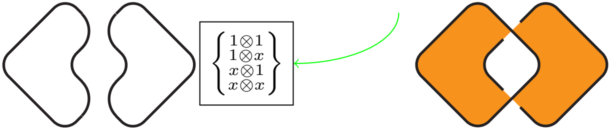

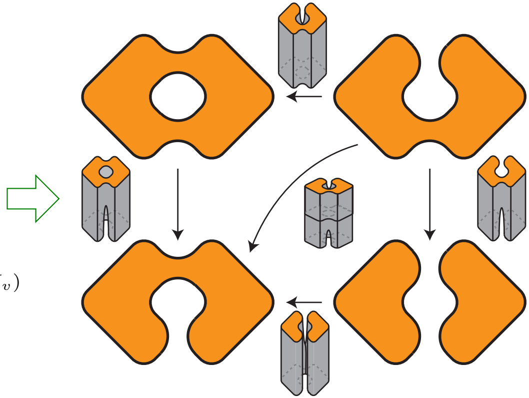

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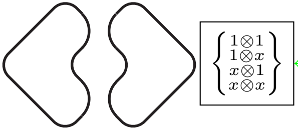

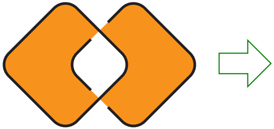

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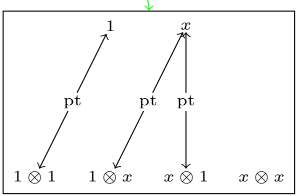

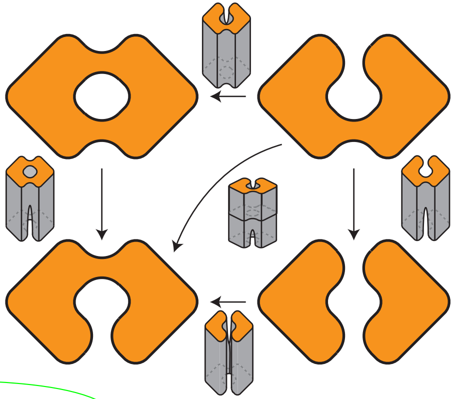

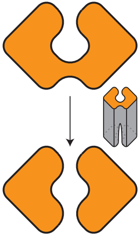

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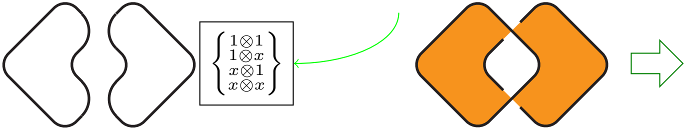

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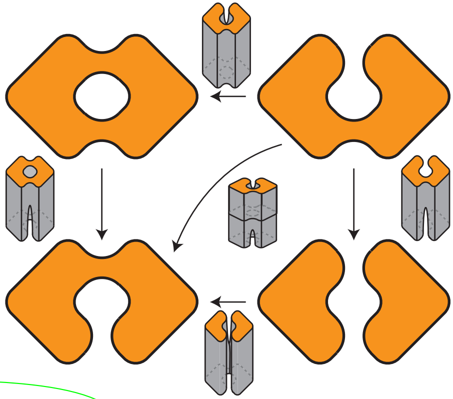
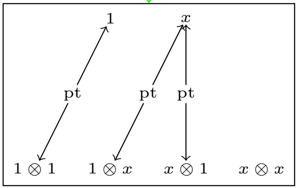
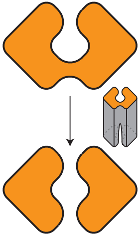
Set of generators of the kernel of

$$\mapsto \; H^1(B) \to H^1(B \cap \{0, 1\} \times \mathbb{R})$$







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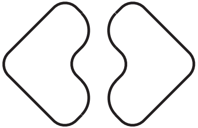

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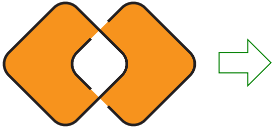

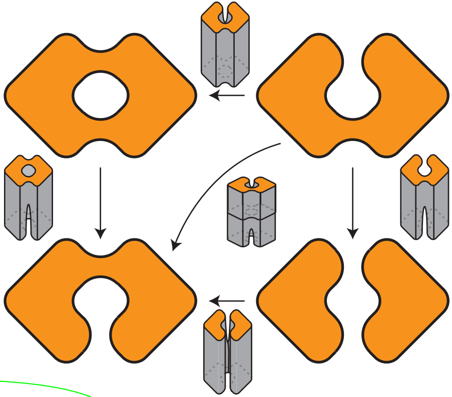

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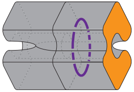

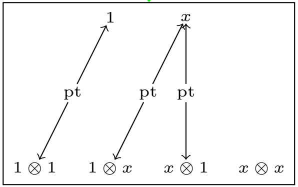

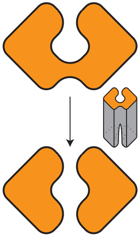

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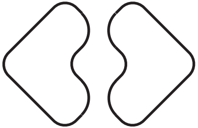

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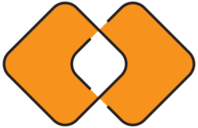

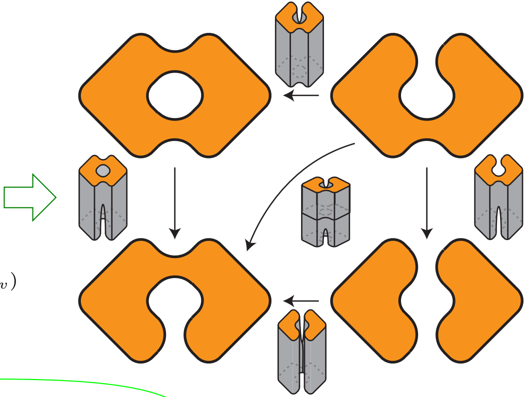

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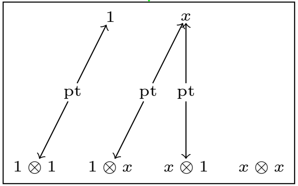

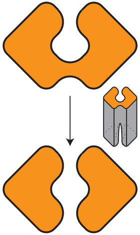

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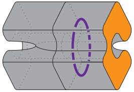

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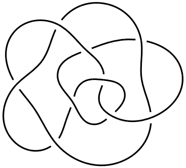

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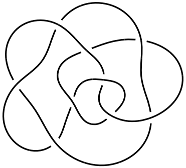

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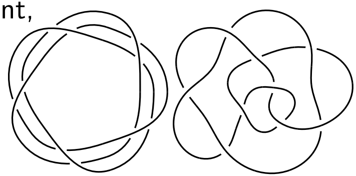

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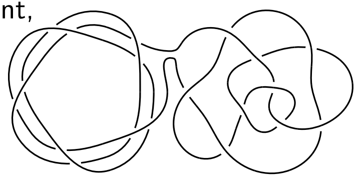

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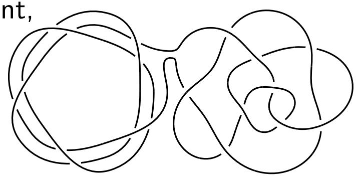

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- (Many other open questions in the written version of this talk.)

Thanks!

• We thank the many colleagues who have helped us learn this material, Mohammed Abouzaid, Ralph Cohen, Chris Douglas, Ciprian Manolescu, and many others. . .

• And most especially our collaborators on this project, Tyler Lawson and Lenhard Ng.

• Thanks also to the organizing committee for inviting us, our hosts for their hospitality, and all of you for listening.