A Heegaard Floer analog of algebraic torsion

Çağatay Kutluhan
Gordana Matić
Jeremy Van Horn-Morris
Andy Wand

June 15, 2015

BACKGROUND AND MOTIVATION

Definition A co-oriented *contact structure* on a 3-manifold M is a 2-plane field ξ defined as the kernel of a *contact 1-form* λ such that $\lambda \wedge d\lambda$ is a volume form. For example, $\lambda = dz - ydx$ on $\mathbb{R}^3$.

Definition An *overtwisted disk* in a contact 3-manifold (M, ξ) is an embedded disk $D \subset M$ such that ξ is transverse to D in $\nu(\partial D)$ and $T\partial D \subset \xi$.

A contact 3-manifold (M, ξ) is called *overtwisted*, if it contains an overtwisted disk. Otherwise, it is called *tight*.

- **Eliashberg:**

$$\{2 - \text{plane fields}\} \sim \{\text{OT contact structures}\}.$$

- Tight contact structures naturally appear on the boundary of symplectic manifolds. (**Eliashberg–Gromov**)

Question How to distinguish tight from OT via Floer homology?

The contact class:

- $c^+(\xi) \in HF^+(-M, \mathfrak{s}_\xi)$ ($\widehat{c}(\xi) \in \widehat{HF}(-M, \mathfrak{s}_\xi)$) by Ozsváth–Szabó. Alternative description of $\widehat{c}(\xi)$ by Honda–Kazez–Matić.
- $\psi(\xi) \in \widecheck{HM}_\bullet(-M, \mathfrak{s}_\xi, c_b, \Lambda_\eta)$ by Kronheimer–Mrowka, and by Taubes.

Theorem (Ozsváth–Szabó, Honda–Kazez–Matić)
If ξ is OT, then $c^+(\xi) = 0$.

(The converse is false.)

Theorem (Kronheimer–Mrowka)
If ξ is symplectically fillable, then $\psi(\xi) \neq 0$.

Theorem (Taubes + Colin–Ghiggini–Honda)

$$\begin{array}{ccc} HF^+(-M, \mathfrak{s}_\xi) & \cong & \widecheck{HM}_\bullet(-M, \mathfrak{s}_\xi) \\ c^+(\xi) & \mapsto & \psi(\xi). \end{array}$$

Factors through Hutchings's embedded contact homology (ECH).

***ECH* preliminaries:** Let Y be a closed orientable 3-manifold. A *stable Hamiltonian structure* on Y is a pair (λ, ω) where

- $\omega \in \Omega^2(Y)$ such that $d\omega = 0$,
- $\lambda \in \Omega^1(Y)$ such that $d\lambda = h\,\omega$ for $h \in C^\infty(Y)$,
- $\lambda \wedge \omega \neq 0$.

The associated *Reeb vector field* R is uniquely defined by

- $\omega(R, \cdot) = 0$,
- $\lambda(R) = 1$.

A closed integral curve γ of R, modulo linear parametrization, is called a *Reeb orbit*.

A Reeb orbit γ is *non-degenerate* if 1 is not an eigenvalue of the linearized return map $P_\gamma : \xi_x \to \xi_x \in SL(2, \mathbb{R})$, where $\xi = ker(\lambda)$.

A Reeb orbit γ is *elliptic* (resp. *hyperbolic*) if $|tr(P_\gamma)| < 2$ (resp. > 2).

Suppose that all Reeb orbits are non-degenerate.

Definition Fix $\Gamma \in H_1(Y)$. An *orbit set* Θ is a finite collection $\{(\gamma, m)\}$ where

- γ are distinct embedded Reeb orbits, and $m \in \mathbb{Z}_+$,
- $\sum m[\gamma] = \Gamma$.

Let (X, ω) be a symplectic cobordism from $(Y_-, \lambda_-, \omega_-)$ to $(Y_+, \lambda_+, \omega_+)$, i.e. $\omega|_{Y_\pm} = \omega_\pm$, and J be an almost complex structure on $\overline{X}$ such that

- ω tames J,
- J is cylindrical on the ends.

Fix $\Theta_\pm \subset Y_\pm$ orbit sets such that $[\Theta_+] = [\Theta_-]$ in $H_1(X)$. Then for any $Z \in H_2(X, \Theta_+, \Theta_-)$, Hutchings defines

$$J_\circ(\Theta_+, \Theta_-, Z) := -c_\tau(Z) + Q_\tau(Z) + \mu'_\tau(\Theta_+) - \mu'_\tau(\Theta_-),$$

and

$$J_\pm(\Theta_+, \Theta_-, Z) := J_\circ(\Theta_+, \Theta_-, Z) \pm |\Theta_+| \mp |\Theta_-|.$$

In particular, if $X = [0,1] \times Y$ for some (Y, λ, ω), then

- J_* depends only on Z,
- J_* is additive, i.e.

$$J_*(\Theta_+, \Theta_-, Z + Z') = J_*(\Theta_+, \Theta, Z) + J_*(\Theta, \Theta_-, Z').$$

Fact If C is an embedded J-holomorphic curve in X with ends at distinct embedded Reeb orbits, then

$$J_\circ(C) = -\chi(C).$$

AN ANALOG OF ALGEBRAIC TORSION (FOLLOWING HUTCHINGS'S RECIPE)

Let (S, ϕ) be an (abstract) open book decomposition for M supporting ξ:

- S compact oriented surface with genus-g and $\textsc{b}$ boundary components,
- $\phi : S \to S$ orientation reversing diffeomorphism.

Let $a = \{a_1, \ldots, a_{\textsc{g}}\}$, where $\textsc{g} = 2g + \textsc{b} - 1$, be a basis of arcs for S.

Then (Σ, α, β) is a Heegaard diagram for M. Suppose that $(\Sigma, \alpha, \beta, z)$ is strongly admissible for $\mathfrak{s}_\xi$ (after an isotopy of ϕ).

Construct $Y \simeq M \# \mathcal{H}_0 \# \mathcal{H}_1 \# \cdots \# \mathcal{H}_{\textsc{g}}$ where $\mathcal{H}_i \simeq S^1 \times S^2$.

There exists a stable Hamiltonian structure (λ, ω) on Y "compatible" with $(\Sigma, \alpha, \beta, z)$.

Fix $\Gamma \in H_1(Y)$ such that $\Gamma|_{H_1(M)} = 0$ and

$$\Gamma \cdot [S^2] = \begin{cases} 0 & \text{in } \mathcal{H}_0 \\ 1 & \text{in } \mathcal{H}_i, i \neq 0. \end{cases}$$

Then for any orbit set $\Theta = \{(\gamma, m)\}$ with $[\Theta] = \Gamma$, γ are **non-degenerate**, and **hyperbolic**. Define

$$\mathcal{Z}_{ech,M} := \{\Theta = \{\gamma\} \mid [\Theta] = \Gamma\}.$$

We have

$$\mathcal{Z}_{ech,M} \simeq \mathcal{Z}_{HF} \times \prod_{i=1}^{\mathrm{G}} \mathbb{Z} \times \mathrm{O},$$

where $\mathrm{O} = \{0, +1, -1, \{+1, -1\}\}$.

Consider the chain complex $(\widehat{ecc}(Y, \lambda, \omega, \Gamma; J), \widehat{\partial}_{ech})$ where $\widehat{ecc}$ is freely generated by $\mathcal{Z}_{ech,M}$.

Let $\Theta_\pm \subset \mathcal{Z}_{ech,M}$ and $C \in \mathcal{M}(\Theta_+, \Theta_-)$ be embedded. Then

$$J_+(C) = -\chi(C) + |\Theta_+| - |\Theta_-|,$$

or

$$J_+(C) = \sum_j (2g(C_j) - 2 + 2|\Theta_{+j}|),$$

where C_j is connected. Hence

- $2 | J_+(C)$,
- $J_+(C) \geq 0$ unless each $g(C_j) = 0$ and $\Theta_{+j} = \emptyset$ for some j.

Instead, work with the subcomplex $\widehat{ecc}_\circ(Y, \lambda, \omega, \Gamma; J)$ freely generated by those $\Theta \in \mathcal{Z}_{ech,M}$ such that

$$\Theta|_{\mathcal{H}_i} = (0,0)$$

for each $i = 1, \ldots, \textsc{g}$. Then, we have

$$\widehat{ecc}_\circ(Y, \lambda, \omega, \Gamma; J) \overset{\textbf{K.–Lee–Taubes}}{\cong} \widehat{CF}(-M, \mathfrak{s}_\xi).$$

On $\widehat{ecc}_\circ(Y, \lambda, \omega, \Gamma; J)$, write

$$\widehat{\partial}_{ech} = \partial_0 + \partial_1 + \cdots + \partial_k + \cdots$$

where ∂_k counts $J_+ = 2k$ curves. Since $\widehat{\partial}_{ech} \circ \widehat{\partial}_{ech} = 0$, and J_+ is additive,

$$\sum_{i+j=l} \partial_i \circ \partial_j = 0$$

for all $l \geq 0$. In particular, $\partial_0 \circ \partial_0 = 0$, $\partial_0 \circ \partial_1 + \partial_1 \circ \partial_0 = 0$,etc.

As a result, there is a spectral sequence $E^*(S,\phi,a;J)$ such that

$$E^{k+1}(S,\phi,a;J) \cong H_*(E^k(S,\phi,a;J),\partial_k),$$

and $E^0(S,\phi,a;J) \cong \widehat{ecc}_\circ(Y,\lambda,\omega,\Gamma;J)$.

Definition Given (S,ϕ,a) and J, let $AT(S,\phi,a;J)$ denote the smallest non-negative integer such that $[\Theta_\xi]=0$ in $E^{k+1}(S,\phi,a;J)$.

(M,ξ) is said to have *algebraic k-torsion* if $AT(S,\phi,a;J)=k$ for some choice of (S,ϕ,a) and J.

Remark If $\widehat{c}(\xi)=0$, then $AT(S,\phi,a;J)<\infty$. The converse is not necessarily true.

Description via Heegaard diagram: Use Lipshitz's reformulation.

$$\begin{array}{rcl} \Theta_\pm & \longleftrightarrow & \mathrm{x}_\pm \\ \mathcal{M}_{I=1}(\Theta_+,\Theta_-) & \overset{1-1}{\longleftrightarrow} & \mathcal{M}_{ind=1}(\mathrm{x}_+,\mathrm{x}_-) \\ C & \longleftrightarrow & C_L. \end{array}$$

$$\begin{array}{rcl} J_+(C) & = & -\chi(C) + |\Theta_+| - |\Theta_-| \\ & = & -\chi(C_L) + \mathrm{G} + |\Theta_+| - |\Theta_-|. \end{array}$$

Menawhile, Lipshitz finds

$$\begin{array}{rcl} \chi(C_L) & = & \mathrm{G} - n_{\mathrm{x}_+}(\mathcal{D}) - n_{\mathrm{x}_-}(\mathcal{D}) + e(\mathcal{D}) \\ ind(C_L) & = & n_{\mathrm{x}_+}(\mathcal{D}) + n_{\mathrm{x}_-}(\mathcal{D}) + e(\mathcal{D}). \end{array}$$

Hence,

$$J_+(C) = 2[n_{\mathrm{x}_+}(\mathcal{D}) + n_{\mathrm{x}_-}(\mathcal{D})] - 1 + |\mathrm{x}_+| - |\mathrm{x}_-|.$$

More generally, for any domain $\mathcal{D}$,

$$J_+(\mathcal{D}) = \mu(\mathcal{D}) - 2e(\mathcal{D}) + |\mathrm{x}_+| - |\mathrm{x}_-|.$$

Theorem If ξ is OT, then (M, ξ) has algebraic 0-torsion.

Proof There exists an open book (S, ϕ) with non-right-veering monodromy, and a basis of arcs a such that

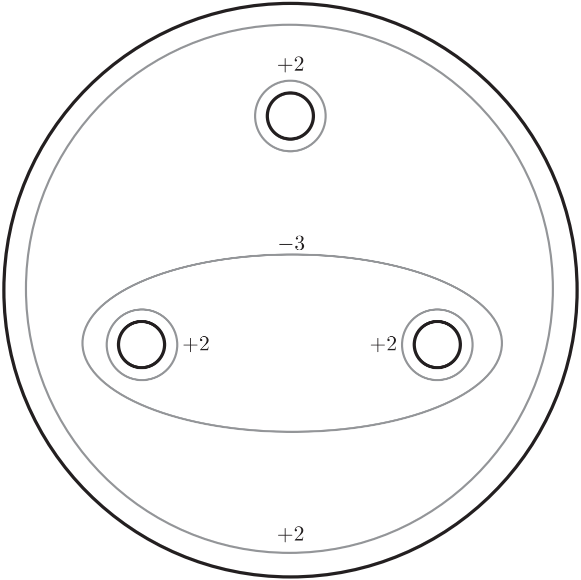
$$\widehat{\partial}_{HF}\mathrm{y} = \mathrm{x}_\xi$$

with one non-trivial index-1 curve, which is represented by a bigon domain $\mathcal{D}$. Then

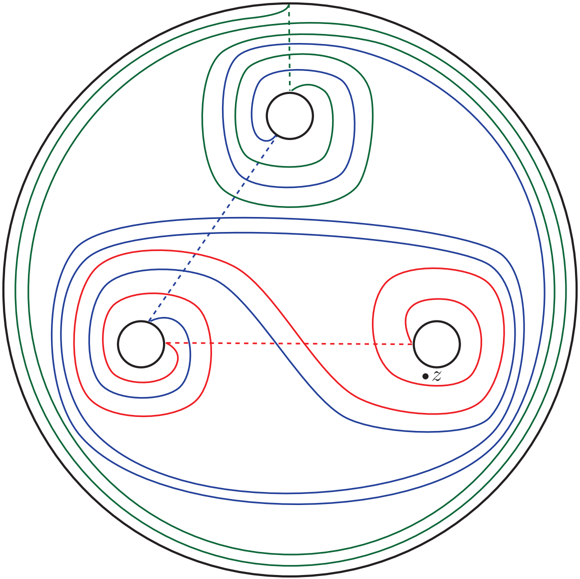
- $n_{\mathrm{y}}(\mathcal{D}) = \frac{1}{4} = n_{\mathrm{x}_\xi}(\mathcal{D})$,
- $|\mathrm{y}| = \mathrm{G} = |\mathrm{x}_\xi|$,

gives $J_+(\mathcal{D}) = 0$.

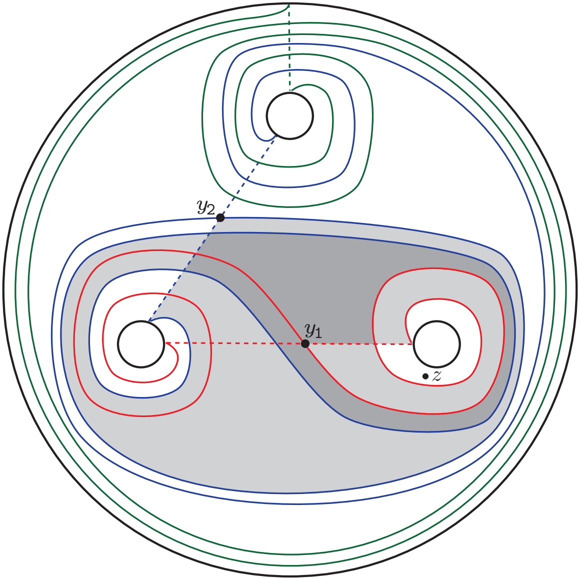
AN EXAMPLE



AN EXAMPLE



AN EXAMPLE



WHAT TO DO NEXT?

Need to show

- J-independence, ✓
- arc basis independence, in progress
- stabilization invariance. ✓given arc basis independence

Question Can AT detect overtwistedness?

Question $AT \leq$ *planar torsion*?

Thanks for your attention!