(1) Prove that the map between the two Mayer-Vietoris sequences in the proof of Poincaré duality commutes up to sign.

$$
\begin{array}{ccccccc}
H^k(U\cup V) & \longrightarrow & H^k(U)\oplus H^k(V) & \longrightarrow & H^k(U\cap V) & \longrightarrow & H^{k+1}(U\cup V) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
(H_c^{n-k}(U\cup V))^* & \longrightarrow & (H_c^{n-k}(U))^*\oplus (H_c^{n-k}(V))^* & \longrightarrow & (H_c^{n-k}(U\cap V))^* & \longrightarrow & (H_c^{n-k-1}(U\cup V))^*
\end{array}
$$

In particular, specify what the signs are for each of the three squares. (You may convince yourself that the Five Lemma works even if the diagram commutes up to sign.)

(2) Let M be a countable disjoint union of oriented manifolds $M_1, M_2, \ldots$.
 (a) Prove $H_c^k(M)$ is the direct sum $\oplus_i H_c^k(M_i)$.
 (b) Prove $H^k(M)$ is the direct product $\prod_i H^k(M_i)$.
 (c) Prove that if the Poincaré duality map $H^k \to (H_c^{n-k})^*$ is an isomorphism for each M_i, then it is an isomorphism for M.
 (d) Prove that the map in the other direction, $H_c^k \to (H^{n-k})^*$ is not an isomorphism in general.

(3) Compute the following using various long exact sequences.
 (a) $H^k(M \setminus \{p\})$ in terms of $H^k(M)$ for a connected manifold M.
 (b) H^k(Möbius strip).
 (c) $H^k(S^1 \times M)$ in terms of $H^k(M)$.
 (d) $H_c^k(S^n \times \mathbb{R}^m)$ and $H^k(S^n \times S^m)$

(4) If a vector field X on an n-dimensional manifold M has an isolated zero at p, show that the index of $-X$ at p is $(-1)^n$ times the index of X at p. Conclude that $\chi(M) = 0$ if n is odd.

(5) Prove that a closed 2-form ω on $S^2 \times S^2$ is exact if and only if $\int_{S^2\times\{p\}} \omega = \int_{\{p\}\times S^2} \omega = 0$ (for any $p \in S^2$).

(6) Let $\omega = (-ydx + xdy)/(x^2+y^2) \in \Omega^1(\mathbb{R}^3 \setminus \text{z-axis})$. For fixed p, q relatively prime positive integers, let $\gamma\colon S^1 \to \mathbb{R}^3 \setminus$ z-axis be the (p,q)-torus knot

$$e^{i\theta} \mapsto \big((\cos(q\theta)+2)\cos(p\theta), (\cos(q\theta)+2)\sin(p\theta), -\sin(q\theta)\big).$$

Compute $\int_\gamma \omega = \int_{S^1} \gamma^*\omega$.