(1) Let $\omega \in \Omega^1(M)$ be such that $\int_c \omega = 0$ for every closed curve $c$ in $M$. Prove $\omega$ is exact (that is, $\omega = df$ for some $f$).

(2) Let $M$ be a connected manifold where every closed curve is smoothly contractible to a point (that is, for any curve $c\colon S^1 \to M$, there is a smooth homotopy $h\colon S^1 \times [0,1] \to M$ with $h|_{\{0\}\times S^1} = c$ and $h|_{\{1\}\times S^1}$ constant.) Prove that $H^1(M) = 0$.

(3) Define the cup product $\cup\colon H^k(M) \otimes H^l(M) \to H^{k+l}(M)$ by

$$[\omega] \cup [\eta] = [\omega \wedge \eta].$$

  (a) Prove it is a well-defined map (that is, it is exact if both are exact, closed if one is closed, and bilinear).
  (b) Prove $\alpha \cup \beta = (-1)^{kl} \beta \cup \alpha$.
  (c) If $f\colon M \to N$, $f^*(\alpha \cup \beta) = f^*(\alpha) \cup f^*(\beta)$.
  (d) Define the cross product $\times\colon H^k(M) \times H^l(N) \to H^{k+l}(M \times N)$ by

$$\alpha \times \beta = \pi_M^*(\alpha) \cup \pi_N^*(\beta),$$

  (where $\pi_M, \pi_N$ are the two projections). If $\Delta\colon M \to M \times M$ is the diagonal inclusion, prove

$$\alpha \cup \beta = \Delta^*(\alpha \times \beta).$$

  (e) If $M, N$ are compact oriented, orient $M \times N$ by declaring $v_1, \ldots, v_m, w_1, \ldots, w_n$ to be positive basis for $T_{(p,q)}(M \times N)$ if $v_1, \ldots, v_m$ is a positive basis for $T_pM$ and $w_1, \ldots, w_n$ is a positive basis for $T_qN$ (this is called the product orientation). Each of the groups $H^m(M), H^n(N), H^{m+n}(M \times N)$ are canonically isomorphic to $\mathbb{R}$ by integrating forms. Prove the following commutes.

$$\begin{array}{ccc} H^m(M) \otimes H^n(N) & \xrightarrow{\ \times\ } & H^{m+n}(M \times N) \\ \downarrow & & \downarrow \\ \mathbb{R} \otimes \mathbb{R} & \xrightarrow{\ a \otimes b \mapsto ab\ } & \mathbb{R} \end{array}$$

(4) If $p, q \in \mathbb{R}^n$ are distinct points, compute the cohomology groups of $\mathbb{R}^n \setminus \{p, q\}$.

(5) Suppose $M$ is a compact oriented manifold whose boundary has two components $\partial_0 M$ and $\partial_1 M$, and let $\iota_0, \iota_1$ be the two inclusions. Let $\alpha$ be a $(p-1)$-form with $\iota_0^*\alpha = 0$ and $\beta$ be a $(n-p)$-form with $\iota_1^*\beta = 0$. Prove

$$\int_M d\alpha \wedge \beta = (-1)^p \int_M \alpha \wedge d\beta.$$

(6) Let $dA$ denote the standard area form on $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$. For which values of $n$ is the form $z^n dA$ exact?