(1) Let Δ be a k-dimensional distribution on an n-dimensional manifold M. Prove that locally near any point p, there are $(n-k)$ 1-forms $\omega_{k+1}, \ldots, \omega_n$ with $\Delta = \ker(\omega_{k+1}) \cap \cdots \cap \ker(\omega_n)$. Further, prove the following are equivalent locally near p.

  (a) Δ is integrable.

  (b) For every i, $d\omega_i(X, Y) = 0$ for any two vector fields X, Y that lie on Δ.

  (c) For every i, there exist 1-forms $\eta_{k+1}, \ldots, \eta_n$ (depending on i) with $d\omega_i = \sum_j \eta_j \wedge \omega_j$.

(2) Consider the 1-form $\omega = xdy - ydx + dz$ on $\mathbb{R}^3$. Prove that $f\omega$ is not closed for any nowhere zero function $f \colon \mathbb{R}^3 \to \mathbb{R}$.

(3) Let ω be a 1-form on M that is non-zero at p. Prove that locally around p, $\omega \wedge d\omega = 0$ if and only if $\omega = \lambda df$ for some smooth functions λ, f defined around p.

(4) Let X be a smooth vector field on a compact manifold M, and let $\phi_t \colon M \to M$ be the flow at time t. Find an explicit chain homotopy between ϕ_1^* and ϕ_0^*, where $\phi_t^* \colon \Omega^*(M) \to \Omega^*(M)$ is the chain map induced by ϕ_t.

(5) (a) The polar angle θ is a function that is locally defined on $\mathbb{R}^2 \setminus 0$, up to some additive constants. Therefore, $d\theta$ is a well-defined 1-form on $\mathbb{R}^2 \setminus 0$. Write down $d\theta$ in local coordinates in terms of dx and dy.

  (b) Let $c \colon [0,1] \to \mathbb{R}^2 \setminus 0$ be $t \mapsto (\cos 2\pi t, \sin 2\pi t)$. Prove $c^*(d\theta) = 2\pi dt$.

  (c) If $\omega = f dt$ is a 1-form on $[0,1]$ with $f(0) = f(1)$, prove that there is a unique number λ such that $\omega - \lambda dt = dg$ for some function g with $g(0) = g(1)$.

  (d) Let $i \colon S^1 \to \mathbb{R}^2 \setminus 0$ be the inclusion. If ω is a 1-form on S^1, prove there is a unique number λ such that $\omega - \lambda i^*(d\theta)$ is exact.