(1) Suppose M is a compact connected 3-manifold, $\omega \in \Omega^1(M)$ is nowhere zero, and $\ker(\omega)$ is an integrable distribution.

  (a) Show that $\omega \wedge d\omega = 0$.

  (b) Use partition of unity to show that there is some $\alpha \in \Omega^1(M)$ with $dw = \alpha \wedge \omega$.

  (c) Show that $d\alpha \wedge \omega = 0$.

  (d) Suppose α' is some other 1-form with $d\omega = \alpha' \wedge \omega$. Show that $\alpha' - \alpha = g\omega$ for some function g.

(2) If $v_1, \dots, v_n$ is a basis for V and $w_i = \sum_j a_{ij} v_j$, show that

$$w_1 \wedge \cdots \wedge w_n = \det(a_{ij}) v_1 \wedge \cdots \wedge v_n.$$

(Therefore, two ordered bases $(v_1, \dots, v_n)$ and $(w_1, \dots, w_n)$ determine the same orientation on V if and only if $v_1 \wedge \cdots \wedge v_n$ is a positive multiple of $w_1 \wedge \cdots \wedge w_n$.)

(3) Prove that n functions $f_1, \dots, f_n \colon M \to \mathbb{R}$ form a coordinate system in a neighborhood of a point p in an n-dimensional manifold M if and only if $df_1 \wedge \cdots \wedge df_n(p) \neq 0$.

(4) Call an element $\omega \in \Lambda^k(V)$ decomposable if $\omega = \phi_1 \wedge \cdots \wedge \phi_k$ for some $\phi_i \in V$.

  (a) Let $\omega \in \Lambda^2(V)$. Prove that there is a basis $\phi_1, \dots, \phi_n$ of V such that

$$\omega = (\phi_1 \wedge \phi_2) + (\phi_3 \wedge \phi_4) + \cdots + (\phi_{2r-1} \wedge \phi_{2r}).$$

  (Hint: If $\omega = \sum_{i<j} a_{ij} \psi_i \wedge \psi_j$ in terms some other basis elements, choose ϕ_1 involving $\psi_1, \psi_3, \dots, \psi_n$ and ϕ_2 involving $\psi_2, \dots, \psi_n$.)

  (b) Show that the r-fold wedge product $\omega \wedge \cdots \wedge \omega$ is decomposable and the $(r+1)$-fold wedge product is zero. Thus r is well-defined; it is called the rank of ω.

  (c) If $\omega = \sum_{i<j} a_{ij} \psi_i \wedge \psi_j$ in terms some other basis elements, and A is the skew symmetric matrix with $A_{ij} = -A_{ji} = a_{ij}$ (and 0 on the diagonal), then the rank of A is twice the rank of ω.

(5) Let M be an n-dimensional manifold and $\omega_1, \dots, \omega_k$ be pointwise linearly independent 1-forms. If $\theta_1, \dots, \theta_p$ are 1-forms (with $p \leq k$) with

$$\sum_{i=1}^{p} \omega_i \wedge \theta_i = 0,$$

prove that there exists smooth functions f_{ij} (with $f_{ij} = f_{ji}$) so that

$$\theta_i = \sum_j f_{ij} \omega_j.$$

(6) Consider the 2-form $\omega = (dx^1 \wedge dx^2) + (dx^3 \wedge dx^4) + \cdots + (dx^{2n-1} \wedge dx^{2n})$ on $\mathbb{R}^{2n}$. Let $f \colon \mathbb{R}^{2n} \to \mathbb{R}$ be some smooth function.

  (a) Show that there is a unique vector field X_f on $\mathbb{R}^{2n}$ so that for any vector field Y on $\mathbb{R}^{2n}$, $df(Y) = \omega(X_f, Y)$.

  (b) Use Cartan's Magic Formula to compute $L_{X_f}(\omega)$.