(1) Show that

$$\Delta = \ker(dx^3 - x^1\, dx^2) \cap \ker(dx^1 - x^4\, dx^2) \subset T\mathbb{R}^4$$

is a smooth 2-dimensional distribution, and determine whether Δ is integrable.

(2) Assume that $N \subset M$ is a codimension one properly embedded submanifold. Show that N can be written as $f^{-1}(0)$, where 0 is a regular value of a smooth function $f\colon U \to \mathbb{R}$ for some open set U containing N, if and only if there is a vector field X on M that is transverse to N.

(3) Let $U \times V \subset \mathbb{R}^m \times \mathbb{R}^n$ (with coordinates t, x) be open, with $0 \in U$. Let $f_i = (f_i^1, \ldots, f_i^n)\colon U \times V \to \mathbb{R}^n$ be smooth functions, for $i = 1, \ldots, m$. Then for every $x \in V$, there is a smooth function

$$\alpha\colon W \to V$$

defined on some open set W, with $0 \in W \subset U$, satisfying

$$\alpha(0) = x$$
$$\frac{\partial \alpha}{\partial t^j}(t) = f_j(t, \alpha(t)) \qquad \forall t \in W,$$

if and only if the following equation holds on some open neighborhood of $(0, x)$ inside $U \times V$:

$$\frac{\partial f_j}{\partial t^i} - \frac{\partial f_i}{\partial t^j} + \sum_k \frac{\partial f_j}{\partial x^k} f_i^k - \sum_k \frac{\partial f_i}{\partial x^k} f_j^k = 0 \qquad \forall i, j \in \{1, \ldots, m\}.$$

Hint: Consider the following m-dimensional distribution on $U \times V$:

$$\Delta_p = \left\{ \sum_i r^i \frac{\partial}{\partial t^i}\Big|_p + \sum_k \Big(\sum_i r^i f_i^k(p) \Big) \frac{\partial}{\partial x^k}\Big|_p : r \in \mathbb{R}^m \right\}.$$

(4) Let θ be the restriction of

$$(x^2\, dx^1 - x^1\, dx^2) + (x^4\, dx^3 - x^3\, dx^4) + \cdots + (x^{2n}\, dx^{2n-1} - x^{2n-1}\, dx^{2n})$$

to the unit sphere $S^{2n-1} \subset \mathbb{R}^n$. Prove that $\ker(\theta)$ is a distribution on S^{2n-1}. Is it integrable?

(5) Let $V = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + R\frac{\partial}{\partial z}$ be a nowhere zero vector field on $\mathbb{R}^3$. Prove that the following are equivalent.
   (a) The orthogonal-to-V plane field is integrable in some neighborhood of 0.
   (b) $V \cdot \mathrm{curl}(V) = 0$ on some neighborhood of 0.
   (c) There exists a nowhere zero real-valued function f on some neighborhood of 0 with $\mathrm{curl}(fV) = 0$.