(1) Construct a vector field on $\mathbb{R}$ whose integral curves through any point are only defined for finite time.

(2) For any (smooth) vector field X and any (smooth) function f on a manifold M, prove

$$L_X(df) = d(L_X f).$$

(3) Consider the following vector fields on $\mathbb{R}^3$:

$$X = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}$$
$$Y = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}$$
$$Z = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$$

For any vector $v = (a, b, c) \in \mathbb{R}^3$, consider the vector field

$$W(v) = aX + bY + cZ.$$

(a) Prove that cross product corresponds to Lie bracket:

$$W(u \times v) = [W(u), W(v)].$$

(b) Geometrically describe the flow of $W(v)$ in terms of v.

(4) For X any vector field and A any (l, k)-tensor field, define the Lie derivate $L_X A$ as another (l, k)-tensor field so that the following hold.

(a) For any two (l, k)-tensor fields A, B, $L_X(A + B) = L_X A + L_X B$.

(b) For any two vector fields X, Y, $L_{X+Y}(A) = L_X A + L_X B$.

(c) For any (l, k)-tensor field A and (l', k')-tensor field A', letting $A \otimes A'$ denote the $(l+l', k+k')$-tensor field, $L_X(A \otimes A') = (L_X A) \otimes A' + A \otimes (L_X A')$.

(d) If $C\colon \mathscr{T}_l^k \to \mathscr{T}_{l-1}^{k-1}$ denote any fixed contraction (of the kl possible ones), $L_X(CA) = C(L_X A)$.

(e) For any vector fields $X_1, \ldots, X_k$ and 1-forms $\omega_1, \ldots, \omega_l$,

$$L_X(A(X_1, \ldots, X_k, \omega_1, \ldots, \omega_l)) = (L_X A)(X_1, \ldots, X_k, \omega_1, \ldots, \omega_l)$$
$$+ \sum_i A(X_1, \ldots, L_X X_i, \ldots, X_k, \omega_1, \ldots, \omega_l) + \sum_i A(X_1, \ldots, X_k, \omega_1, \ldots, L_X\omega_i, \ldots, \omega_l).$$

Furthermore, if

$$A = \sum A_{i_1 \ldots i_k}^{j_1 \ldots j_l}\, dx^{i_1} \ldots dx^{i_k} \frac{\partial}{\partial x^{j_1}} \cdots \frac{\partial}{\partial x^{j_l}}$$

in local coordinates, write down $L_X A$ in local coordinates.

(5) If $f\colon M \to N$ is an onto map that is regular (Df surjective) everywhere, and M is compact, prove that f is a smooth fiber bundle. That is, each point $q \in N$ has an open neighborhood U and a diffeomorphism $f^{-1}(U) \cong f^{-1}(q) \times U$ commuting with the maps to U. (Hint 1: This is a local statement, so enough to consider $N = \mathbb{R}^n$. Hint 2: Do the case $N = \mathbb{R}$ first. Hint 3: For the case $N = \mathbb{R}$, construct the diffeomorphism $f^{-1}(U) \cong f^{-1}(q) \times U$ by flowing along some vector field.)