(1) Let M_n be the space of $n \times n$ real matrices, and let M_n^k be the subspace of all matrices of rank k. Prove that M_n^k is a submanifold of M_n. (Hint 1: Do the cases $k = n-1, n-2$ first. Hint 2: Fix some $k \times k$ minor and consider the subspace of M_n where this minor has non-zero determinant.)

(2) The n-dimensional torus T^n is defined to be $\mathbb{R}^n/\mathbb{Z}^n$, i.e., for any $x, y \in \mathbb{R}^n$, $x \sim y$ iff $x - y \in \mathbb{Z}^n$. Let $\alpha, \beta \colon \mathbb{R}^n \to \mathbb{R}$ be two nowhere zero functions such that (i) $\alpha(x) = \alpha(y)$ and $\beta(x) = \beta(y)$ if $x - y \in \mathbb{Z}^n$ and (ii) $\alpha(x)/\beta(x)$ is an irrational constant. Then the vector field

$$\alpha(x)\frac{\partial}{\partial x^1} + \beta(x)\frac{\partial}{\partial x^2}$$

on $\mathbb{R}^n$ descends to a vector field X on T^n. Find all functions $f \colon T^n \to \mathbb{R}$ such that $Xf = 0$.

(3) An n-manifold is called parallelizable if it has n vector fields which are linearly independent at each point.

(a) Prove that S^3 is parallelizable.

(b) Prove that $S^1 \times S^2$ is parallelizable.

(In general, all orientable 3-manifolds are parallelizable, but that is slightly harder.)

(c) Prove that $S^1 \times S^n$ is parallelizable. (Hint: S^n is the unit sphere in $\mathbb{R}^{n+1}$, and $\mathbb{R}^{n+1}$ is parallelizable.)

(4) Let M be a connected smooth manifold. Show that for any points $x, y \in M$, there is a diffeomorphism $f \colon M \to M$ such that $f(x) = y$.

(5) View S^n as the unit sphere in $\mathbb{R}^{n+1}$; the restriction of the standard metric on $\mathbb{R}^{n+1}$ makes S^n a Riemannian manifold. Consider the stereographic projection

$$x \colon U = S^n \setminus (0, \dots, 0, 1) \to \mathbb{R}^n, \qquad (p_1, \dots, p_n, p_{n+1}) \mapsto \left(\frac{p_1}{1 - p_{n+1}}, \dots, \frac{p_n}{1 - p_{n+1}}\right).$$

Write down the metric on U explicitly as $\sum_{i,j} g_{ij}\, dx^i \otimes dx^j$ in terms of these local coordinates.