(1) View S^2 as the unit sphere in $\mathbb{R}^3$.

(a) Let $p_0 \in S^2$ be the point $(0,0,1)$. If $SO(3)$ is the special orthogonal group, define $f\colon SO(3) \to S^2$ by $f(A) = A(p_0)$. Prove that this is a smooth fiber bundle with fiber $SO(2) \cong S^1$.

(b) The tangent bundle T_*S^2 carries a Riemannian metric coming from the dot product in $\mathbb{R}^3$. Let ST_*S^2 be its unit sphere bundle, which is a fiber bundle over S^2, whose fiber over each point are the length-one tangent vectors over that point. Construct a diffeomorphism between ST_*S^2 and $SO(3)$ as fiber bundles over S^2.

(c) Consider the Gauss map $S^2 \to G_2(\mathbb{R}^3) \cong \mathbb{RP}^2$. (Here $G_2(\mathbb{R}^3)$ is the Grasmannian of 2-dimensional planes in $\mathbb{R}^3$.) Prove that this is the orientation double cover. (The Gauss map $S^1 \to G_1(\mathbb{R}^2) \cong \mathbb{RP}^1$ is *not* the orientation double cover, so explain what is different in this situation.)

(2) Let $f\colon M \to N$, and suppose (x, U) and (y, V) are coordinate systems around p and $f(p)$, respectively.

(a) If $g\colon N \to \mathbb{R}$, prove

$$\frac{\partial(g \circ f)}{\partial x^i}(p) = \sum_j \frac{\partial g}{\partial y^j}(f(p))\frac{\partial(y^j \circ f)}{\partial x^i}(p).$$

(b) Prove that

$$f_*(\frac{\partial}{\partial x^i}|_p) = \sum_j \frac{\partial(y^j \circ f)}{\partial x^i}(p)\frac{\partial}{\partial y^j}|_{f(p)}.$$

and, more generally, express $f_*(\sum_i a^i \partial/\partial x^i|_p)$ in terms of $\partial/\partial y^j|_{f(p)}$.

(c) Show that

$$(f^* dy^j)(p) = \sum_i \frac{\partial(y^j \circ f)}{\partial x^i}(p) dx^i(p).$$

(d) More generally, express

$$f^*(\sum_{j_1,\ldots,j_k} a_{j_1,\ldots,j_k} dy^{j_1} \otimes \cdots \otimes dy^{j_k})$$

in terms of the dx^i.

(3) (a) Construct isomorphisms $\phi_V\colon V \to V^{**}$ for all finite dimensional vector spaces V so that for any linear map $f\colon V \to W$, the following commutes.

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ {\scriptstyle\phi_V}\downarrow & & \downarrow{\scriptstyle\phi_W} \\ V^{**} & \xrightarrow{f^{**}} & W^{**} \end{array}$$

(b) Prove that there does not exist isomorphisms $\phi_V\colon V \to V^*$ for all finite dimensional vector spaces V so that for any linear map $f\colon V \to W$, the following commutes.

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ {\scriptstyle\phi_V}\downarrow & & \downarrow{\scriptstyle\phi_W} \\ V^{*} & \xleftarrow{f^{*}} & W^{*} \end{array}$$

(c) Construct isomorphisms $\phi_V\colon V \to V^*$ for all finite dimensional inner-product vector spaces V so that for any linear map $f\colon V \to W$ between inner product spaces that preserves inner products, the following commutes.

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ {\scriptstyle\phi_V}\downarrow & & \downarrow{\scriptstyle\phi_W} \\ V^{*} & \xleftarrow{f^{*}} & W^{*} \end{array}$$

(4) Prove that the tangent bundle and the cotangent bundle of a manifold are always isomorphic (although not canonically).