(1) Construct a smooth manifold structure on the Grassmannian $G_n(\mathbb{R}^k)$, the set of all n-dimensional subspaces of $\mathbb{R}^k$. Construct the canonical n-dimensional vector bundle over it (as a smooth vector bundle, not just the fiber over each point).

(2) Let M be a compact n-dimensional manifold and $f\colon M \to \mathbb{R}^n$ a smooth map. Prove that f is singular (that is, df has rank less than n) somewhere.

(3) A Riemannian structure on a smooth manifold is a choice of a positive definite inner product $\langle , \rangle_p$ on each tangent space T_pM, which is smooth in the sense that whenever X and Y are two smooth vector bundles, $\langle X, Y \rangle$ is smooth. Prove that there is a Riemannian structure on every smooth manifold. (Hint: Use partition of unity.)

(4) Prove that there are exactly two isomorphism classes of line bundles (one-dimensional vector bundles) over S^1. Which of them is the tangent bundle T_*S^1?

(5) If $A \subset M$ is a closed submanifold and $U \supset A$ is any open neighborhood, and $f\colon A \to \mathbb{R}$ is a smooth function, prove that there is a smooth function $g\colon M \to \mathbb{R}$ with $g|_A = f$ and $g = 0$ outside U.

(6) Consider the maximal atlas on $\mathbb{R}$ containing the function $t \mapsto t^3$. Prove that it produces a smooth structure on $\mathbb{R}$ that is distinct but diffeomorphic to the standard smooth structure on $\mathbb{R}$ (the standard smooth structure comes from the maximal atlas containing the function $t \mapsto t$).