Differential Geometry of Curves and Surfaces by Do Carmo.
Elements of Differential Geometry by Millman and Parker.
(The notation and conventions are different in the two books. In class, we will follow conventions from Do Carmo and notation from Millman and Parker.)

Background.

(1) Derivative. Given differentiable $\mathbb{R}^m \xrightarrow{f} \mathbb{R}^n$, and a point $a \in \mathbb{R}^m$, the derivative df_a is a linear map $\mathbb{R}^m \to \mathbb{R}^n$. If we fix coordinates $x_1, \ldots, x_m$ and $y_1, \ldots, y_n$, that fixes a basis, and can represent df_a by a $n \times m$ matrix whose entries are $\partial y_j / \partial x_i$. So $df \colon \mathbb{R}^m \to \mathbb{R}^{mn}$ is the derivative function.
(2) C^0 means continuous. If df exists and $df \colon \mathbb{R}^m \to \mathbb{R}^{mn}$ is C^{k-1}, then f is C^k. Inductive definition. C^k means k times differentiable and k^{th} derivative is continuous. C^∞ is smooth, infinitely differentiable.
(3) Chain rule. Consider $\mathbb{R}^m \xrightarrow{f} \mathbb{R}^n \xrightarrow{g} \mathbb{R}^p$, and fix coordinates $x_1 \ldots, x_m, y_1 \ldots, y_n, z_1, \ldots, z_p$, and a starting point $a \in \mathbb{R}^m$. Then $a \mapsto f(a) \mapsto g(f(a))$, and we have linear maps $\mathbb{R}^m \xrightarrow{df_a} \mathbb{R}^n \xrightarrow{dg_{f(a)}} \mathbb{R}^p$. Then as linear maps $d(g \circ f)_a = dg_{f(a)} \circ df_a$. As matrices $d(g \circ f)_a = dg_{f(a)} df_a$. In terms of matrix entries, $\frac{\partial z_k}{\partial x_i} = \sum_{j=1}^n \frac{\partial z_k}{\partial y_j} \frac{\partial y_j}{\partial x_i}$. Double index notation, latter is written $\sum \frac{\partial z_k}{\partial y_j} \frac{\partial y_j}{\partial x_i}$.

Basic curves in 3D.

(1) Fix interval $I \subset \mathbb{R}$, open or closed (could be $I = \mathbb{R}$) Curve $\alpha \colon I \to \mathbb{R}^3$, different from its image. Think of particle traveling in space; image is the path traced out.
(2) Velocity $\alpha' = d\alpha/dt \colon I \to \mathbb{R}^3$. We view $\alpha(t)$ as a point in $\mathbb{R}^3$, but $\alpha'(t)$ as a 3D vector, drawn starting at $\alpha(t)$. This is also the tangent vector to the curve. Speed is length of the vector $|\alpha'|$.
(3) To avoid sharp corners, we will assume curves are regular, that is $\alpha'(t) \neq 0 \forall t$. The path $\vee$ can be the image of a C^∞ (smooth) path, but it has a sharp corner, and doesn't look smooth. If we impose regularity, then well-defined tangent direction at each point, and so no more sharp corners.
(4) For regular curves, unit tangent $T(t) = \alpha'(t)/|\alpha'(t)|$, unit vector in the tangent direction. Velocity α' was extrinsic (depends on how a particle travels a given path), but $T(t)$ is clearly intrinsic (depends only on the path and direction on travel).
(5) Reparametrization. Consider two intervals I, J, and $h \colon I \to J$ a bijection, so that both h and h^{-1} are C^3. Then if $\alpha \colon I \to \mathbb{R}^3$ and $\beta \colon J \to \mathbb{R}^3$ are related by $\alpha = \beta \circ h$ (equivalently $\beta = \alpha \circ h^{-1}$), then one is a reparametrization of the other. Both of them have the same image, represent the same path in space, but traveled differently (with different speed, etc).
(6) Reparametrization again, $h \colon I \to J$, $g \colon J \to I$, $g \circ h = \mathrm{Id}_I$, $h \circ g = \mathrm{Id}_J$, both C^k. If α, β related by reparametrization, takes regular curves to regular curves. Clearly C^k to C^k. $\frac{\partial \beta}{\partial s}|_a = \frac{\partial \alpha}{\partial t}|_{g(a)} \frac{\partial g}{\partial s}|_a$, so just need to show $\frac{\partial g}{\partial s}|_a \neq 0$. Chain rule again on $h \circ g = \mathrm{Id}_J$.
(7) Alternate description of reparametrization. $h \colon I \to J$ onto, $h \in C^k$, and $h'(t) \neq 0 \forall t$. Injective by Mean Value Theorem; indeed two cases, $h' > 0$ or $h' < 0$ (strictly increasing or strictly decreasing). So if $g = h^{-1}$, why $g \in C^k$?
(8) Review Inverse Function theorem $f \colon \mathbb{R}^n \to \mathbb{R}^n$. If df_a $(n \times n)$ non-singular, then locally has inverse, and inverse is also C^k. In case $n = 1$ (like now), has global inverse (as we saw), but for general n, not. Example: $\mathbb{C} \to \mathbb{C}, z \mapsto e^z$, so $f \colon \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \mapsto (e^x \cos y, e^x \sin y)$, compute $|df| = e^x \neq 0$, but no global inverse since not injective.
(9) If two curves α, β, related by reparametrization $\mathrm{Im}(\alpha) = \mathrm{Im}(\beta)$, but possibly traveled with different speeds. If $h' > 0$, then same orientation (direction of travel), otherwise opposite.
(10) What about converse? If two regular curves have same image, they are related by reparametrization, since regular curves have a unique canonical reparametrization.

Arc length parametrization.

(1) Assume regular curve α. Arc length. Consider starting point $\alpha(t_0)$. Arc length is distance traveled from t_0 to t, integral of speed $s(t) = \int_{t_0}^t |\alpha'(t)| dt$. This is intrinsic, depends only on the starting point, and the direction of travel, independent of parametrization.
(2) Say $h = s$ is the arc-length parametrization, and $J = \mathrm{Im}(h)$. Why is $h \colon I \to J$ a reparametrization? If g is the inverse, define the arc-length parametrization $\beta(s) = \alpha \circ g(s)$.

(3) What is the velocity (which was extrinsic) under arc-length parametrization? $\frac{d\beta}{ds} = \frac{d\alpha}{dt}\frac{dg}{ds} = \frac{d\alpha}{dt}/\frac{dh}{dt} = \frac{d\alpha}{dt}/|\frac{d\alpha}{dt}| = T$, the unit tangent. In particular, unit speed. So arc-length parametrization is canonical: start at the starting point, and travel with unit speed in the given orientation.
(4) Example. Consider the curve $\alpha(t) = (r\cos t, r\sin t, ht)$. Assume $r, h > 0$ constants. Do arc length reparametrization of helix. Useful variable $\omega = \frac{1}{\sqrt{r^2+h^2}}$.
(5) Draw $(r\cos t^2, r\sin t^2, ht^2)$. Different image from the helix. Sharp turn, not regular at $t = 0$.
(6) Draw $(r\cos t^3, r\sin t^3, ht^3)$. Same image as the helix, but still not a reparametrization since not regular at $t = 0$. The change of variable functions are t^3 (which is C^∞) and $t^{1/3}$ (which is not C^1).
(7) Although arc length parametrization always possible for unit speed curves, very hard in practice. Consider the parabola $y = x^2/2$, find arc length parametrization starting at $(0, 0)$. First find regular parametrization $\alpha(t) = (t, t^2/2)$, then $s(t) = \int_0^t \sqrt{1+t^2}dt = \frac{1}{2}(t\sqrt{1+t^2}+\ln(t+\sqrt{1+t^2}))$. (Do $t = \tan\theta$, then integrate $\sec^3\theta$ by parts.) Then arc length parametrization is $\alpha(t(s))$, but impossible to write down the inverse function $t(s)$.

Curvature and torsion.

(1) Nevertheless, only consider unit speed curves $\alpha(s)$ from now. Velocity is unit tangent: $\alpha'(s) = T(s)$. Acceleration $\alpha''(s) = T'(s)$ measures rate of change of T, so change in the direction of travel. So curvature $\kappa(s) = |T'(s)|$.
(2) If $\kappa(s) \neq 0$, then $N(s) = T'(s)/\kappa(s)$, the direction of the rate of change. So $T' = \kappa N$. Note, N only defined when curvature is non-zero.
(3) N is perpendicular to T, since for any unit vector v, v' is perpendicular to v (differentiate $v \cdot v$).
(4) Do helix example. $\kappa = r\omega^2 = \frac{r}{r^2+h^2}$, and N points inwards towards the axis. Special case, $h = 0$, circle of radius r. Curvature is $1/r$, smaller circles have larger curvature.
(5) From now on assume $\kappa(s) \neq 0 \forall s$. Needed to make sense of N. Then we get a right-handed orthonormal basis T, N, B, where $B = T \times N$; explain.
(6) Orthonormal frame, right-handed (positive) vs left-handed (negative). (i, j, k), (k, j, i), (j, k, i) are positive. Cyclic (even) permutations. Form a basis. Any vector can be written uniquely, and the coefficients are given by dot products.
(7) Let's keep differentiating. N' is perpendicular to N, so $N' = aT + bB$. Since N, T orthogonal, $a = -\kappa$. Let $b = -\tau$, torsion. (Osculating plane spanned by T, N; the plane spanned by N, B is normal plane, and the plane spanned by T, B is rectifying plane. τ is the rate of rotation of B in the normal plane.)
(8) One more derivative: $B' = \tau N$. (B' perpendicular to B, and its coefficient at N, T is negative of the coefficients of N', T' at B.)
(9) The data (κ, τ, T, N, B) called Frenet-Serret apparatus. Very important to remember, only considering the Frenet-Serret apparatus for unit speed curve. If not-unit speed, then reparametrize (which might be hard) and then consider this data. Moreover, N, B, τ only defined if $\kappa \neq 0$.
(10) Calculate (κ, τ, T, N, B) for the helix $(r\cos t, r\sin t, ht)$, $\tau = h\omega^2 = \frac{h}{r^2+h^2}$.
(11) If $\kappa = 0$ on an interval, then T constant, so linear. If $\kappa = 0$ at an isolated point, this story not valid, and the osculating plane can change drastically, $\alpha(t) = (t, e^{-1/t^2}, 0)$ or $(t, 0, e^{-1/t^2})$.
(12) If $\kappa > 0, \tau = 0$ on an interval, then B constant, so particle travels in the plane perpendicular to B.
(13) Frenet-Serret equation

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}$$

Proof: Decompose a vector along orthonormal basis by dot products. Treating T, N, B as row vectors. So the equation is $3 \times 3 = (3 \times 3)(3 \times 3)$. Also note the matrix is skew-symmetric.

Fundamental theorem of curves.

(1) Picard's theorem. $I \subset \mathbb{R}$ open interval around 0, $c \in \mathbb{R}^n$, $A\colon \mathbb{R}^n \times I \to \mathbb{R}^n$, uniformly (in time) bounded partial derivatives wrt space coordinates. Then unique $\alpha\colon I \to \mathbb{R}^n$ with $\alpha(a) = c$ and $d\alpha/dt = A(\alpha(t), t)$. Particle traveling in $\mathbb{R}^n$, initial condition specified, and velocity specified depending on its position and time.
(2) Counterexample $x' = x^{1/3}, x(0) = 0$. No solutions for $t < 0$, two solutions $x = \pm\sqrt{(2t/3)^3}$ for $t > 0$.
(3) Outline of proof; iterations of integrals. $\phi_n(t) = c + \int_a^t A(\phi_{n-1}(t), t)dt$.

(4) Fundamental theorem of curves. Any regular curve with $\kappa > 0$ is uniquely determined by κ and τ. More precisely: $0 \in I \subset \mathbb{R}$ (I open interval), $\overline{\kappa} \in C^1(I)$ with $\overline{\kappa}(s) > 0 \forall s$, $\overline{\tau} \in C^0(I)$, $x_0 \in \mathbb{R}^3$, D, E, F right-handed orthonormal basis of $\mathbb{R}^3$. Then unique C^3 unit-speed curve $\alpha\colon I \to \mathbb{R}^3$ with $\alpha(0) = x_0$, $(T(0), N(0), B(0)) = (D, E, F)$, $\kappa(s) = \overline{\kappa}(s)$, and $\tau(s) = \overline{\tau}(s)$.

(5) Proof. Picard uniquely specifies the frame $\overline{T}, \overline{N}, \overline{B}$. The function $A\colon \mathbb{R}^9 \times I \to \mathbb{R}^9$ is given by

$$A(\overline{T}_1, \overline{T}_2, \overline{T}_3, \overline{N}_1, \overline{N}_2, \overline{N}_3, \overline{B}_1, \overline{B}_2, \overline{B}_3, s) = \begin{pmatrix} 0 & \overline{\kappa}(s) & 0 \\ -\overline{\kappa}(s) & 0 & -\overline{\tau}(s) \\ 0 & \overline{\tau}(s) & 0 \end{pmatrix} \begin{pmatrix} \overline{T}_1 & \overline{T}_2 & \overline{T}_3 \\ \overline{N}_1 & \overline{N}_2 & \overline{N}_3 \\ \overline{B}_1 & \overline{B}_2 & \overline{B}_3 \end{pmatrix}.$$

So satisfies the Lipshitz condition. For instance

$$\frac{\partial A}{\partial \overline{T}_1} = \begin{pmatrix} 0 & 0 & 0 \\ -\overline{\kappa}(s) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since $\overline{\kappa}, \overline{\tau} \in C^0$, bounded (on any closed interval in I, which is enough). Define $\alpha(s) = x_0 + \int_0^s \overline{T}(s) ds$.

(6) Show $\overline{T}, \overline{N}, \overline{B}$ positive orthonormal. Consider Picard's theorem again with six variables $\overline{x} \cdot \overline{y}$, $x, y \in \{T, N, B\}$. Uniqueness of solution forces orthonormality. Positivity is forced by continuity of $[\overline{T}, \overline{N}, \overline{B}]$—triple scalar product.

(7) Immediate that α is unit speed. To show $\alpha \in C^3$, need $T \in C^2$, so $\overline{\kappa} N \in C^1$, which we have. (This is exactly where we needed $\overline{\kappa}$ to be C^1—to ensure the curve is C^3, which is needed in order to define torsion.)

(8) Easy to show $\overline{x} = x$ for $x = T, N, \kappa, B, \tau$ in this order.

(9) If τ, $\kappa > 0$ constant, then circular helix. Proof. Just show it satisfies the equation, with $r = \frac{\kappa}{\kappa^2+\tau^2}$, $h = \frac{\tau}{\kappa^2+\tau^2}$. ($\tau < 0$ right-handed, $\tau > 0$ left-handed, $\tau = 0$ circle.)

Non-unit speed curves.

(1) Regular but not (necessarily) unit speed curve, $\alpha(s(t))$. Derivatives wrt s are prime, wrt t are dots. Chain rule, $\dot{x} = x'\dot{s}$. Let speed $v = \dot{s} = |\dot{\alpha}|$. Running example $\alpha(t) = (t, t^2, t^3)$.

(2) Rest follows by just differentiating $\dot{\alpha} = vT$. ($T = \dot{\alpha}/v$.)

(3) $\ddot{\alpha} = \dot{v}T + v\dot{T} = \dot{v}T + v^2\kappa N$, $\dot{\alpha} \times \ddot{\alpha} = v^3\kappa B$. ($\kappa = |\dot{\alpha} \times \ddot{\alpha}|/v^3$, $B = \dot{\alpha} \times \ddot{\alpha}/\kappa v^3$, $N = B \times T$.)

(4) $\dddot{\alpha} = \ddot{v}T + \dot{v}v\kappa N + (\dot{\overline{v^2\kappa}})N - v^3\kappa^2 T - v^3\kappa\tau B$, $[\dot{\alpha}, \ddot{\alpha}, \dddot{\alpha}] = -(v^3\kappa)^2\tau$. ($\tau = -[\dot{\alpha}, \ddot{\alpha}, \dddot{\alpha}]/(v^3\kappa)^2$.)

(5) Frenet-Serret equations.

$$\begin{pmatrix} \dot{T} \\ \dot{N} \\ \dot{B} \end{pmatrix} = \begin{pmatrix} v & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & v \end{pmatrix} \begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & \kappa v & 0 \\ -\kappa v & 0 & -\tau v \\ 0 & \tau v & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}$$

Rotation index.

(1) Define T, n for plane C^2 curves, by setting $b = \hat{k}$ and $n = b \times T$. Explain $n = \pm N$, if the latter is defined, and explain the sign. (n is obtained by rotating T by 90°.) Define planar curvature k_P by $T' = k_P n$, which can be negative (turning left vs turning right.) k_P measures the rate of change of direction of T.

(2) More precise version of the previous statement. Write $T(s) = (\cos(\theta(s)), \sin(\theta(s)))$, explain how to make sense locally. If $T(s)$ is C^1, so is $\theta(s)$. Then $k_P(s) = \theta'(s)$.

(3) Homework problem. Find curve with $\kappa(s) = \frac{1}{1+s^2}$, $\tau(s) = 0$, $x_0 = 0$, $(D, E, F) = (i, j, k)$. Planar motion in xy plane. $T(s) = (\cos\theta(s), \sin(\theta(s)))$, and $\kappa(s) = k_P(s) = \theta'$. (Note $\kappa(s)$ never zero, so binormal B throughout $\hat{k}$ constant, same as b.)

(4) After fixing $\theta(0)$, globally define $\theta(s)$ by cutting interval into small pieces where $\theta(s)$ doesn't change by more than 180° (points up, down, right, or left, only). Alternate definition $\theta(s) = \theta(0) + \int_0^s k(s) ds$.

(5) Periodic (closed) curve, $\alpha(s + L) = \alpha(s) \forall s$, period is smallest such L (which is length of the curve $\alpha\colon [0, L] \to \mathbb{R}^2$. Index of periodic curve $\frac{1}{2\pi}\int_0^L k ds$.

(6) Index is total change of θ divided by 2π since $\theta' = k_P$. Interesting index examples: circle oriented both ways, clover leaf.

(7) Simple closed curve, no self-intersection, $\alpha(s) \neq \alpha(t) \forall s \neq t \in [0, L)$. Total index is ± 1.

(8) Jordan curve theorem (very hard to prove). Complement of simple closed curve has outside (the non-compact region containing ∞) and inside.

(9) Simple closed curve is positively oriented if the inside region is on the left.
(10) Index of positively oriented simple closed curve is 1. After translating if necessary, let $\alpha(0)$ be the lowest point of the curve. That is, if $\alpha(s) = (x(s), y(s))$, it is minimum of $y(s)$, defined since $[0, L]$ is compact. The tangent there is horizontal since $y'(s) = 0$.
(11) Define (u, v) maps to the angle of the vector $\alpha(v) - \alpha(u)$ on the triangle $0 \le u \le v \le L$. Limiting case when $u = v$ is the tangent line. This angle well-defined up to multiples of 2π, but can be well-defined globally by cutting into small triangles, and fixing a starting value, say $(0, 0) \mapsto 0$. Need triangle is simply connected.
(12) The theta difference along the hypotenuse is 2π times the index, but along but along each of the other two sides is π, since $\alpha(0) = \alpha(L)$ was the lowest point, so can never turn more than π. (That is, $(0, L) \mapsto \pi$ and $(L, L) \mapsto 2\pi$.)

Coordinate patches.

(1) Just like curves were function $\alpha\colon I \to \mathbb{R}^3$, surface is a function $x\colon U \to \mathbb{R}^3$, where U is an open set in $\mathbb{R}^2$.
(2) What is an open set in $\mathbb{R}^n$? For every point $p \in U$, there exists some $r > 0$, so that $B_r(p) \subset U$. Informally, does not contain any point on the boundary. Open intervals are open sets in $\mathbb{R}$. Complement of open is called closed. (Lots of sets are neither open nor closed.)
(3) Checking whether some function is C^k is local. So makes sense to talk about whether x is C^k. We assume $x \in C^k$, and $k \ge 1$ (usually $k \ge 3$).
(4) We also assume x is injective. That is, for simple surfaces like this (also called coordinate patches), do not allow self-intersections.
(5) Finally a regularity condition, just like we assumed $\alpha'(s) \ne 0$ for curves to have a well-defined unit tangent T. Let u^1, u^2 be the coordinates on $\mathbb{R}^2$, and let $x_i = \frac{\partial x}{\partial u^i}$ (which are 3-dimensional vectors). We assume $x_1 \times x_2 \ne 0$ everywhere. That is, not only are each x_1 and x_2 non-zero, they are linearly independent.
(6) Geometric meaning. Hold u_2 constant, change u_1, get a straight line in U, produces a curve in the surface, its tangent vector is x_1. Similarly, hold u_1 constant, change u_2, get a different curve, its tangent vector is u_2. These curves are called parametric curves. The regularity condition says x_1 and x_2 are linearly independent (always draw them starting at $x(p)$), that is, they span a 2-dimensional plane, which is called the tangent plane at p (or $x(p)$). What is the unit normal? ($n = \frac{x_1 \times x_2}{|x_1 \times x_2|}$.)
(7) Example is a graph. Let $f\colon U \to \mathbb{R}$, $f \in C^k$. Then graph is a function $U \to \mathbb{R}^3$, $(u^1, u^2) \mapsto (u^1, u^2, f(u^1, u^2))$. Why is this a simple surface? Check C^k, injective, and the regularity condition.
(8) Example of an example, graph of $f(u^1, u^2) = \sqrt{1 - (u^1)^2 - (u^2)^2}$. What is the domain? Unit disk, but we want open domain, so open unit disk. The surface is the upper hemisphere.
(9) Another way to visualize the sphere, spherical coordinates. Consider the surface

$$(\theta, \phi) \mapsto (\cos\theta\cos\phi, \sin\theta\cos\phi, \sin\phi).$$

What are the parametric curves? Latitudes (circles, usually not great) and longitudes (half great circles). We need injective, so what is a good domai? $(0, 2\pi) \times (-\pi/2, pi/2)$. What is the image? (Unit sphere minus the poles, as well as the prime meridian.) Finally, check regularity: $|x_\theta \times x_\phi| = \cos\phi > 0$. What is unit normal? Can find it geometrically as well since it points outwards.
(10) Next we study reparametrization. Completely analogous to curves. Consider surfaces $x\colon U \to \mathbb{R}^3$ and $y\colon V \to \mathbb{R}^3$. Reparametrization are C^k functions $h\colon U \to V, g\colon V \to U$, with $h \circ g = \mathrm{Id}_V, g \circ h = \mathrm{Id}_U$ and $x = y \circ h$ (equivalently $y = x \circ g$).
(11) For curves, it was equivalent to saying $h'(s) \ne 0 \forall s$. What about dh_p, for $p \in U$? What is dh_p? Linear map $\mathbb{R}^2 \to \mathbb{R}^2$, represented by 2×2 matrix $(\frac{\partial u^i}{\partial u^j})$ since we have basis. This matrix is called the Jacobian. It is non-singular. Has an inverse $df_{h(p)}$ by chain rule, $dg_{h(p)} \circ dh_p = \mathrm{Id}$.
(12) But unlike for curves, this is not enough to check, since inverse function theorem is only local. So to check reparametrization, need to check h is bijective (with inverse say g), $h \in C^k$, dh_p is non-singular everywhere. No need to check $g \in C^k$, since that is a local statement and follows from inverse function theorem. (This will actually be automatic if both x and y are regular, by Implicit function theorem.)

(13) Also unlike curves, no canonical parametrization. So to define any invariants of surfaces, first need to parametrize, then define it in terms of that parametrization, and then check independence of parametrization.

(14) We have so far: Tangent plane, unit normal. Check independence under reparametrization. Draw picture of $x_i = \frac{\partial x}{\partial u^i}$ and $y_j = \frac{\partial y}{\partial v^j}$ on the same surface. Why is plane spanned by y_1, y_2 same as plane spanned by x_1, x_2? How are they related?

(15) Since $y = x \circ g$, chain rule,

$$y_i = \frac{\partial y}{\partial v^i} = \sum \frac{\partial x}{\partial u^j}\frac{\partial u^j}{\partial v^i} = \sum x_j \frac{\partial u^j}{\partial v^i}$$

(in the double index summation notation). In terms of matrices $2 \times 3 = (2 \times 2)(2 \times 3)$,

$$\begin{pmatrix} y_1 & y_2 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} \frac{\partial u^1}{\partial v^1} & \frac{\partial u^1}{\partial v^2} \\ \frac{\partial u^2}{\partial v^1} & \frac{\partial u^2}{\partial v^2} \end{pmatrix}$$

(16) To check same tangent plane, just need to check same normal vector. So

$$y_1 \times y_2 = \det(J)(x_1 \times x_2).$$

Recall $\det(J) \neq 0$, so simultaneously checks that the condition of regularity is preserved under transformation ($x_1 \times x_2 \neq 0$ implies $y_1 \times y_2 \neq 0$), as well as, unit normal preserved up to sign (depending on whether $\det(J) > 0$ or < 0), and hence the tangent plane is also preserved.

(17) What is a tangent vector X (usually capital letters)? By definition it is a linear combination of x_1 and x_2. To preserve double index notation, write $\vec{X} = \sum_i X^i \vec{x}_i$.

(18) Alternate description, tangent vectors are velocity vectors of curves through that point. That is, fix $p \in U$, let $q = x(p)$ be its image on the surface, and let $\alpha \colon I \to U$ be a curve through p (with $\alpha(0) = p$). Then composite $x \circ \alpha$ is a curve on the surface through q. We claim, the velocity of this curve is a tangent vector. Chain rule,

$$\frac{d(x \circ \alpha)}{dt}|_{t=0} = \frac{\partial x}{\partial u^1}|_p \frac{du^1}{dt}|_{t=0} + \frac{\partial x}{\partial u^2}|_p \frac{du^2}{dt}|_{t=0} = x_1 \frac{du^1}{dt} + x_2 \frac{du^2}{dt}$$

is a linear combination of x_1 and x_2.

Surfaces.

(1) Definition of C^k surface, example is $S^2 \subset \mathbb{R}^3$. Recall simple surface, but S^2 cannot be covered by a single coordinate chart. So need a bunch.

(2) Digression about topology. Need a notion of open sets, example $\mathbb{R}^n$. Subset of a topological space is a topological space. Do examples of open sets in S^2. One extreme example is whole S^2 is open in S^2, but not in $\mathbb{R}^3$.

(3) Surface is a subset $S \subset \mathbb{R}^3$, and a collection (could be infinite, indeed usually uncountable) of coordinate charts $x \colon U \to \mathbb{R}^3$ (which are all C^k, injective, regular), satisfying the following.

(4) Each x maps to S (that is $x(U) \subset S$), and the map $x \colon U \to S$ is a homeomorphism. (This means $x^{-1} \colon x(U) \to U$ is also continuous.)

(5) For each $p \in S^2$, there is some coordinate chart x with $p \in x(U)$. That the coordinate charts cover the whole surface.

(6) Finally, the different charts are compatible. So if we have two charts $x \colon U \to S$, $y \colon V \to S$, we get a bijection $y^{-1} \circ x \colon x^{-1}(x(U) \cap y(V)) \to y^{-1}(x(U) \cap y(V))$. We require this to be a C^k reparametrization, that is, each of the maps $y^{-1} \circ x$ and $x^{-1} \circ y$ are C^k.

(7) This compatibility allows us to define tangent space T_pS—a two dimensional vector space—at each point $p \in S$ (usually drawn at p). Choose a coordinate chart x that covers p (which exists), define tangent space for x as the linear span of x_1 and x_2, and then check it is well-defined, that is, independent of choices. So if y is another coordinate chart also covering p, then the span of y_1 and y_2 is the same space. That is true, since they are reparametrizations of $x(U) \cap y(V)$.

(8) Example of surface: S^2. We have already seen many parametrizations of S^2, like upper hemisphere (recall, $x(u^1, u^2) = (u^1, u^2, \sqrt{1 - (u^1)^2 - (u^2)^2})$ or spherical coordinates. Can cover S^2 with six such hemispheres, but let's do something different.

(9) Stereographic projection from the unit sphere minus north pole $(0,0,1)$ to the plane (draw picture); inverse is a coordinate chart. Equation of straight line from $(v^1,v^2,0)$ to $(0,0,1)$ is $(tv^1,tv^2,1-t)$, so when on sphere $t=\frac{2}{(v^1)^2+(v^2)^2+1}$, so coordinate chart given by

$$y(v^1,v^2)=\big(\frac{2v^1}{(v^1)^2+(v^2)^2+1},\frac{2v^2}{(v^1)^2+(v^2)^2+1},\frac{1-(v^1)^2-(v^2)^2}{(v^1)^2+(v^2)^2+1}\big)$$

which is C^∞ since we are not dividing by 0.

(10) So x and y cover S^2, so just need to check overlap condition. The domain and range of $y^{-1}\circ x$ are the punctured disk and complement of unit disk.

(11) Write down $y^{-1}\circ x$, $(u^1,u^2)\mapsto\big(\frac{u^1}{1-\sqrt{1-(u^1)^2-(u^2)^2}},\frac{u^1}{1-\sqrt{1-(u^1)^2-(u^2)^2}}\big)$, (figure out x^{-1} by drawing the straight line $(tp,tq,1+t(q-1))$ from $(0,0,1)$ to the point (p,q,r) on the sphere), from punctured disk and its inverse $x^{-1}\circ y$, $(v^1,v^2)\mapsto\big(\frac{2v^1}{(v^1)^2+(v^2)^2+1},\frac{2v^2}{(v^1)^2+(v^2)^2+1}\big)$, from the complement of the disk. Check both are C^∞, so reparametrization, so defined S^2 as a surface.

First fundamental form.

(1) First recall linear map, $V\cong\mathbb{R}^n$ is n-dimensional vector space (for us, $n=2$), $f\colon V\to V$ is linear if $f(\lambda u+\mu v)=\lambda f(u)+\mu f(v)\forall\lambda,\mu,u,v$. If we choose a basis $\{x_1,\dots,x_n\}$, then f determined by its value on x_i, so if $f(x_i)=\sum f_i^j x_j$, then f is uniquely determined by matrix (f_i^j). Conversely, a matrix A determines a linear map $v\mapsto Av$.

(2) Bilinear map is something completely different. Map $g\colon V\times V\to\mathbb{R}$ which is linear in each factor, $g(\lambda u+\mu v,w)=\lambda g(u,w)+\mu g(v,w)$ and similarly. If we choose a basis, also determined by a matrix $g_{ij}=g(x_i,x_j)$. Conversely, matrix B determines a bilinear map $(u,v)\mapsto u^TBv$.

(3) Matrix represents bilinear form if both indices subscripts, but a linear map if one index subscript one index superscript.

(4) Symmetric, positive definite ($g(v,v)>0\forall v\neq 0$) bilinear map is called an inner product. Corresponding matrix is symmetric and positive definite.

(5) Back to surfaces. Given a surface S and point $p\in S$, have defined 2-dimensional tangent space T_pS. One of the first examples of an abstract vector space, does not come with natural basis. (We can choose a coordinate chart x, and will get a basis x_1,x_2, but some other coordinate chart will give some other basis.) Nevertheless has an inner product, first fundamental form, $g(u,v)=\langle u,v\rangle=I(u,v)=u\cdot v$. We defined without parametrizing, so property of surface (independent of parametrization). If we fix a patch, g represented by matrix (g_{ij}) where $g_{ij}=x_i\cdot x_j$.

(6) Example $S=\mathbb{R}^2$, parametrize as $x(u^1,u^2)=(u^1,u^2)$, g is the identity matrix. Identical for the cylinder $x(\theta,z)=(\cos\theta,\sin\theta,z)$.

(7) The sphere S^2 with spherical coordinates $x(\theta,\phi)=(\cos\theta\cos\phi,\sin\theta\cos\phi,\sin\phi)$, $0<\theta<2\pi$, $-\pi/2<\phi<\pi/2$. We get

$$g=\begin{pmatrix}\cos^2\phi & 0\\ 0 & 1\end{pmatrix}$$

(8) We are studying local properties of surfaces. Local means depends only on a neighborhood, like C^k, tangents, normals, curvature, most things in the course so far. Indeed only global thing so far is total index of simple closed curves. Property means independent of parametrization. Example, tangent space T_pS, normal $\pm n$, first fundamental form g. When did we check g is independent of parametrization? (g was defined without using parametrization.) Of course, if we choose a parametrization, g is given a matrix (g_{ij}) with $g_{ij}=x_i\cdot x_j$, and if we change parametrization, we will get a different matrix which is related to this by multiplying with J^t and J on two sides.

(9) Local properties are classified into intrinsic or extrinsic. Intrinsic means can access them if you are a two dimensional creature living on the surface, cannot look up. (In Flatland, the protagonist is a triangle living in $\mathbb{R}^2$). So T_pS is intrinsic, since tangent vectors are just directions of travel, and length is how fast you are traveling. So is g, since $u\cdot v=|u||v|\cos\theta$, and we can measure lengths and angles. But normal is extrinsic, since we cannot look up.

(10) Technical definition of intrinsic, anything that depends only on g (since lengths and angles are all that we can measure); everything else is extrinsic. So how do we prove $\pm n$ is extrinsic?

(11) Parametrize cylinder, $x(u^1, u^2) = (\cos u^1, \sin u^1, u^2)$, compute first fundamental form, same as that of plane. So locally looks the same (isometric) to the plane. (Globally different, since on the cylinder you can keep travelling in certain directions and come back.) But for the plane n is constant, but for the cylinder n changes, so n is extrinsic.

(12) Is S^2 locally isometric to $\mathbb{R}^2$? Different first fundamental form $\begin{pmatrix} \cos^2\phi & 0 \\ 0 & 1 \end{pmatrix}$, so can say nothing yet. (Different parametrization might have produced the same first fundamental form.) Indeed, we will see later that S^2 is not locally isometric to the plane—that is, if you are a 2-dimensional creature living on S^2, just by measuring lengths and angles locally, you should be able to tell. This is how people figured out that Earth is not flat.

Curves on surfaces

(1) Particle travelling on S^2, longitude, latitude at time t is given by $(a(t), b(t))$. What is distance travelled in time 0 to t? Reasonable real-world problem, we know GPS coordinates. So $(a(t), b(t))$ gives a curve $\alpha\colon I \to \mathbb{R}^2$ to (θ, ϕ) plane, and $x\colon \mathbb{R}^2 \to S^2$ is the spherical coordinates chart, and the curve is $x \circ \alpha$. So

$$\begin{aligned} v &= \frac{d(x\circ\alpha)}{dt} = x_1\frac{da}{dt} + x_2\frac{db}{dt} \\ |v|^2 = g(v,v) &= g_{11}\big(\frac{da}{dt}\big)^2 + 2g_{12}\frac{da}{dt}\frac{db}{dt} + g_{22}\big(\frac{db}{dt}\big)^2 \\ &= \cos^2(b(t))\big(\frac{da}{dt}\big)^2 + \big(\frac{db}{dt}\big)^2 \end{aligned}$$

and distance travelled is $\int_0^t v dt$.

(2) An aside. Notice, for us, usually g is given by a diagonal matrix, that is x_1 and x_2 are perpendicular. This is because we are choosing parametrizations carefully. This is not true for arbitrary parametrizations.

(3) So that was an example of a curve on a surface. So we have a curve $\alpha = (\alpha^1, \alpha^2)\colon I \to U \subset \mathbb{R}^2$, and a coordinate patch $x\colon U \to S \subset \mathbb{R}^3$, and by composing we get a curve $x \circ \alpha\colon I \to \mathbb{R}^3$ on the surface S.

(4) We have already seen the tangent vector to the curve $\sum x_i \frac{d\alpha^i}{dt}$, a linear combination of x_1 and x_2, is in T_pS.

(5) As before, from now on, we will impose the curve is unit speed, and write s instead of t. That is a messy condition,

$$\sum g_{ij}\frac{d\alpha^i}{dt}\frac{d\alpha^j}{dt} = \begin{pmatrix} \frac{d\alpha^i}{dt} & \frac{d\alpha^j}{dt} \end{pmatrix} \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} \frac{d\alpha^i}{dt} \\ \frac{d\alpha^j}{dt} \end{pmatrix} = 1.$$

The unit tangent then is given by $T = \frac{d(x\circ\alpha)}{ds} = \sum x_i \frac{d\alpha^i}{ds}$.

(6) Next for unit speed curves, we can compute curvature $kN = dT/ds$. If n is unit normal to surface, T is unit normal to curve, then set $S = n \times T$; n, T, S form a positive orthonormal frame. The curvature kN has no component along T, so we can write $\kappa N = \kappa_g S + \kappa_n n$.