This is a non-collaborative closed-book exam. You are not allowed to use books, notes, or any electronic devices (such as calculators, phones, computers) during the exams. There are a total of three problems and a total of eight pages. You need to justify all answers. Write your solutions in the space below the questions. If you need more space use the back of the page. Do not forget to write your name in the space above.

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1. CIRCULAR HELIX

For any fixed real number $h$, consider the circular helix
\[ \alpha(t) = \left( \cos \left( \frac{t}{\sqrt{1 + h^2}} \right), \sin \left( \frac{t}{\sqrt{1 + h^2}} \right), \frac{ht}{\sqrt{1 + h^2}} \right), \quad t \in \mathbb{R}, \]
as a curve on the cylinder $x^2 + y^2 = 1$.

1.1. (5pts) Prove that it is a geodesic.

1.2. (5pts) Prove that any geodesic on the cylinder passing through the point $(1, 0, 0)$ is either of the above form, or the straight line $(1, 0, t), t \in \mathbb{R}$ (which is the limiting case when $h \to \infty$), or those curves with opposite orientation.

**Solution**

1.1. It is unit speed since
\[ \alpha'(t) = \left( -\frac{1}{\sqrt{1 + h^2}} \sin \left( \frac{t}{\sqrt{1 + h^2}} \right), \frac{1}{\sqrt{1 + h^2}} \cos \left( \frac{t}{\sqrt{1 + h^2}} \right), \frac{h}{\sqrt{1 + h^2}} \right) \]
has length one. Moreover,
\[ \kappa N = \alpha''(t) = \left( -\frac{1}{1 + h^2} \cos \left( \frac{t}{\sqrt{1 + h^2}} \right), -\frac{1}{1 + h^2} \sin \left( \frac{t}{\sqrt{1 + h^2}} \right), 0 \right) \]
points towards the axis, which is the same direction as the normal to the cylinder. So the entire curvature is in the normal direction, and there is no geodesic curvature. Therefore, $\alpha$ is a geodesic.

1.2. By the existence and uniqueness theorem for geodesics, through the point $(1, 0, 0)$, there exists one and only one geodesic in each tangent direction. At $t = 0$ when $\alpha$ passes through $(1, 0, 0)$, the tangent direction is
\[ \alpha'(0) = \left( 0, \frac{1}{\sqrt{1 + h^2}}, \frac{h}{\sqrt{1 + h^2}} \right); \]
and for the same curve with opposite orientation, the tangent direction is
\[ \left( 0, -\frac{1}{\sqrt{1 + h^2}}, -\frac{h}{\sqrt{1 + h^2}} \right); \]
and finally, in the limiting cases when $h \to \infty$, the tangent directions are
\[ (0, 0, 1) \text{ and } (0, 0, -1). \]
These directions cover all the possible tangent directions to the cylinder at $(1, 0, 0)$ (since the two-dimensional tangent plane to the cylinder at $(1, 0, 0)$ is spanned by $(0, 1, 0)$ and $(0, 0, 1)$), and therefore, these are all the geodesics through the point $(1, 0, 0)$. 
2. The donut

Consider the surface $T$ consisting of points of the form
\[ x(u, v) = ((2 + \cos(u)) \cos(v), (2 + \cos(u)) \sin(v), \sin(u)), \]
for $u, v \in \mathbb{R}$.

2.1. (5pts) For any real numbers $A, B$, consider
\[ x|(A, A + 2\pi) \times (B, B + 2\pi), \]
that is, the function $x(u, v)$ restricted to $A < u < A + 2\pi$ and $B < v < B + 2\pi$. Prove that each of these functions is a coordinate patch on the surface $T$.

2.2. (5 pts) Prove that these coordinate patches are compatible with one another, and they cover the whole surface $T$.

2.3. (5pts) What is the smallest number of such patches that you need to cover $T$? Justify your answer.

Solution

2.1. The function $x$ is clearly a $C^\infty$ function.

To check regularity,
\[ x_u = (-\sin(u) \cos(v), -\sin(u) \sin(v), \cos(u)) \]
\[ x_v = (-2 \cos(u)) \sin(v), (2 \cos(u)) \cos(v), 0), \]
and so,
\[ x_u \times x_v = -2 \cos(u)(\cos(u) \cos(v), \cos(u) \sin(v), \sin(u)(\cos^2(v) + \sin^2(v))) \]
\[ = -2 \cos(u)(\cos(u) \cos(v), \cos(u) \sin(v), \sin(u)), \]
which is non-zero since
\[ |x_u \times x_v| = |2 + \cos(u)| \sqrt{\cos^2(u) \cos^2(v) + \cos^2(u) \sin^2(v) + \sin^2(u)} = 2 + \cos(u) > 1. \]

To check that $x$, once restricted to $(A, A + 2\pi) \times (B, B + 2\pi)$, is injective, note that if a point $(a, b, c)$ is on the surface $T$, then $\cos(u), \sin(u), \cos(v), \sin(v)$ are all determined in terms of $a, b, c$,
\[ \cos(u) = \sqrt{a^2 + b^2} - 2 \quad \sin(u) = c \]
\[ \cos(v) = \frac{a}{2 + \cos(u)} = \frac{a}{\sqrt{a^2 + b^2}} \quad \sin(v) = \frac{b}{2 + \cos(u)} = \frac{b}{\sqrt{a^2 + b^2}}; \]
and therefore, either of $u$ and $v$ is determined up to adding multiples of $2\pi$. In particular, there is at most one $u \in (A, A + 2\pi)$
and at most one \( v \in (B, B + 2\pi) \) with the above specified values of \( \cos(u), \sin(u), \cos(v), \sin(v) \).

Finally, to show that each of these charts is proper, we need to show that the inverse maps are continuous. The above shows that \( \cos(u), \sin(u), \cos(v), \sin(v) \) are all continuous functions of \( a, b, c \), and therefore, the \( u \in (A, A + 2\pi) \) determined by \( \cos(u) \) and \( \sin(u) \) and the \( v \in (B, B + 2\pi) \) determined by \( \cos(v) \) and \( \sin(v) \) are continuous functions of \( a, b, c \) as well.

2.2. Let’s just show that these patches are compatible with one another. In the next part, we will show that the surface can be covered by just three of these.

So let \( x|_{(A,A+2\pi)\times(B,B+2\pi)} \) and \( x|_{(C,C+2\pi)\times(D,D+2\pi)} \) be two patches. We need to show that they are compatible. For clarity, let us use different names. Let \( f = x|_{(A,A+2\pi)\times(B,B+2\pi)} \) and let \( g = x|_{(C,C+2\pi)\times(D,D+2\pi)} \). Let \( (u,v) \) denote the variables for \( f \) and let \( (r,s) \) denote the variables for \( g \). Let \( R \) denote the open square \( (A,A + 2\pi) \times (B,B + 2\pi) \) in the \( uv \)-plane and let \( S \) denote the open square \( (C,C + 2\pi) \times (D,D + 2\pi) \) in the \( rs \)-plane.

The images of the patches intersect in \( f(R) \cap g(S) \). Its preimages in the two coordinate planes are \( R' := f^{-1}(g(S)) \) and \( S' := g^{-1}(f(R)) \). We need to produce a bijective \( C^k \) function, whose inverse is also \( C^k \), from \( R' \) to \( S' \), so that \( f \) is the composition of \( g \) and this function.

Since \( g(S) \) just misses two circles in the surface \( T \), \( R' \) covers the entire open square \( R \) in the \( uv \)-plane, except possibly two lines. Similarly, \( S' \) covers the entire open square \( S \) in the \( rs \)-plane except possibly two lines. So depending on the values of \( A, B, C, D \), the open set \( R' \) in the \( uv \)-plane (and the open set \( S' \) in the \( rs \)-plane) is a disjoint union of one, two, or four open rectangles.

Since the values of the trigonometric functions are determined, the transformation from \( R' \) to \( S' \) must satisfy
\[
\cos(r) = \cos(u) \quad \cos(s) = \cos(v) \quad \sin(r) = \sin(u) \quad \sin(s) = \sin(v),
\]
and so the transformation from \( (u, v) \) to \( (r, s) \) must be of the form
\[
r = u + (\text{integer})2\pi \quad s = v + (\text{integer})2\pi.
\]
These integers need not be constant on the entire set \( R' \), but they will be constant on each of the (up to four) components of \( R' \), and so the transformation function will still be \( C^\infty \). Similarly, the inverse function is also \( C^\infty \).

The following diagram illustrates the transformation function when \( R = (-3\pi, -\pi) \times (-\pi/2, 3\pi/2) \) in the \( uv \)-plane and \( S = (4\pi, 6\pi) \times (3\pi, 5\pi) \) in the \( rs \)-plane; in this case, each of \( R' \) and \( S' \) consists of four open
rectangles, colored orange, blue, pink, and green. The transformation sends orange to orange by translating by \((6\pi, 2\pi)\), blue to blue by translating by \((8\pi, 2\pi)\), pink to pink by translating by \((8\pi, 4\pi)\), and green to green by translating by \((6\pi, 4\pi)\).

2.3. A single patch \(x|_{(A,A+2\pi)\times(B,B+2\pi)}\) misses the union of the 2 circles:
\[
\left((2 + \cos(A)) \cos(v), (2 + \cos(A)) \sin(v), \sin(A)\right) \quad \text{and} \quad \left((2 + \cos(u)) \cos(B), (2 + \cos(u)) \sin(B), \sin(u)\right).
\]

Adding a second patch \(x|_{(C,C+2\pi)\times(D,D+2\pi)}\) still misses at least two points:
\[
\left((2 + \cos(A)) \cos(D), (2 + \cos(A)) \sin(D), \sin(A)\right) \quad \text{and} \quad \left((2 + \cos(C)) \cos(B), (2 + \cos(C)) \sin(B), \sin(C)\right).
\]
(It misses more if \(A = C\) or \(B = D\).)

Therefore, we need at least three patches. The following three patches suffice.
\[
x|_{(-\pi,\pi)\times(-\pi,\pi)}, x|_{(0,2\pi)\times(0,2\pi)}, x|_{(-\pi/2,3\pi/2)\times(-\pi/2,3\pi/2)}.
\]
3. THE SPHERE

(5pts) Let Earth be the unit sphere \(x^2 + y^2 + z^2 = 1\). What is the distance (on Earth) from Los Angeles (34°N, 118°W) to New York City (40°N, 74°W)?

Do not simplify: leave the answer in terms of trigonometric and inverse trigonometric functions, like \(\arctan(\cos(34°) + \sin(74°))\).

Hint: What are the \((x, y, z)\)-coordinates of Los Angeles and New York?

**Solution**

If a point has latitude \(a\) and longitude \(b\), then its polar angle \(\theta\) is \(b\), and its azimuthal angle \(\phi\) (angle from positive z-axis) is \(90° - a\), and so its Cartesian coordinates are

\[
\begin{pmatrix}
\cos \theta \sin \phi, \\
\sin \theta \sin \phi, \\
\cos \phi
\end{pmatrix} = \begin{pmatrix}
\cos(b) \cos(a), \\
\sin(b) \cos(a), \\
\sin(a)
\end{pmatrix}.
\]

Therefore, the Cartesian coordinates of Los Angeles and New York are given by

\[
P = \begin{pmatrix}
\cos(-118°) \cos(34°), \\
\sin(-118°) \cos(34°), \\
\sin(34°)
\end{pmatrix}
\quad \text{and}
\quad Q = \begin{pmatrix}
\cos(-74°) \cos(40°), \\
\sin(-74°) \cos(40°), \\
\sin(40°)
\end{pmatrix}.
\]

The distance is given by the length the shortest geodesic. Since geodesics are great circles, it is the length of the shorter arc of the great circle connecting \(P\) and \(Q\). The length of an arc is radius (which is 1 since it is a great circle) times the angle subtended at the center of the circle (which is the origin \(O\) since the circle is a great circle). So the distance is simply \(\alpha\) where \(\alpha\) is the angle between the vectors \(\vec{OP}\) and \(\vec{OQ}\).

Using dot product to calculate \(\alpha\),

\[
\cos \alpha = \frac{\vec{O}P \cdot \vec{O}Q}{||\vec{O}P|| ||\vec{O}Q||} = \frac{\vec{O}P \cdot \vec{O}Q}{||\vec{O}P|| ||\vec{O}Q||}
\]

\[
= \left(\cos(-118°) \cos(-74°) + \sin(-118°) \sin(-74°)\right) \cos(34°) \cos(40°) + \sin(34°) \sin(40°)
\]

\[
= \cos(-74° + 118°) \cos(34°) \cos(40°) + \sin(34°) \sin(40°)
\]

\[
= \cos(44°) \cos(34°) \cos(40°) + \sin(34°) \sin(40°),
\]

the required distance is

\[
\alpha = \arccos\left(\cos(44°) \cos(34°) \cos(40°) + \sin(34°) \sin(40°)\right).
\]