MAT 120A
FINAL
DEC 5, 2016
3–6PM

NAME:

- This is a non-collaborative closed-book exam. You are not allowed to use books, notes, or any electronic devices (such as calculators, phones, computers) during the exams.
- There are a total of five problems and a total of fourteen pages.
- You need to justify all answers.
- Do not forget to write your name in the space above.

1	10	
2	10	
3	10	
4	10	
5	60	
Total	100	

1. Distances

1.1. (4pts) Let M be a surface in $\mathbb{R}^3$ that intersects a plane P in a curve γ, and assume M is symmetric with respect to P. Prove that γ, when parametrized by arc length, is a geodesic of M.

1.2. (6pts) Let M be the surface $x^4 = y + z^4$. Calculate the distances between the following pairs of points on M. (You may leave the answers as definite integrals.)

1.2.1. $A = (1, 0, -1)$ and $B = (-2, 0, 2)$.

1.2.2. $C = (0, -1, 1)$ and $D = (0, -1, -1)$.

1.2.3. $E = (1, 1, 0)$ and $F = (2, 16, 0)$.

2. Non-positivity of Gaussian curvature

(Recall: Gaussian curvature is the product of the maximum and minimum normal curvature.)

2.1. (3pts) Prove that the Gaussian curvature at a point p on a surface M is non-positive if and only if there is a curve on M through p whose *normal* curvature at p is zero.

2.2. (2pts) In fact, prove that if the Gaussian curvature at a point p on a surface M is non-positive, then there is a curve on M through p whose *total* curvature at p is zero.

2.3. (2pts) Assume that a surface M contains a straight line L. Prove that at each point of L, the Gaussian curvature of M is non-positive.

2.4. (3pts) Give an example of a surface M containing a straight line L, so that the Gaussian curvature of M at each point of L is strictly negative.

3. Lines of curvature

(10pts) Consider a coordinate chart $\mathbf{x}(u^1, u^2)$ and assume that each point the two principal curvatures are unequal. Prove that the u^i-curves are lines of curvature if and only if $L_{12} = g_{12} = 0$.

4. Isoperimetric inequality

(10pts) What is the maximum area that can be enclosed by a closed curve of length 1 on the cylinder $x^2 + y^2 = 1$?

5. Torus

Consider the surface T consisting of points of the form

$$\mathbf{x}(u^1, u^2) = \big((2+\cos(u^1))\cos(u^2), (2+\cos(u^1))\sin(u^2), \sin(u^1)\big),$$

for $u^1, u^2 \in \mathbb{R}$.

5.1. (6pts) For any real numbers A, B, consider

$$\mathbf{x}|_{(A,A+2\pi)\times(B,B+2\pi)},$$

that is, the function $\mathbf{x}(u^1, u^2)$ restricted to $A < u^1 < A+2\pi$ and $B < u^2 < B+2\pi$. Prove that each of these functions is a proper coordinate patch on the surface T, that these patches are compatible with one another, and that they cover the entire surface T.

5.2. (6pts) Compute $\mathbf{x}_1$ and $\mathbf{x}_2$, which form a basis for the tangent space of T. Compute the unit normal vector $\mathbf{n}$.

5.3. (6pts) Compute the matrix (g_{ij}) which represents the first fundamental form on the tangent space in terms of the basis from Problem 5.2. Check that it is positive definite. Compute the inverse matrix (g^{ij}).

5.4. (6pts) Compute the matrix (L_{ij}) which represents the second fundamental form on the tangent space in terms of the basis from Problem 5.2. Is it positive definite?

5.5. (6pts) Let $\vec{X} = 5\mathbf{x}_1 + 3\mathbf{x}_2$ and $\vec{Y} = 2\mathbf{x}_1 - \mathbf{x}_2$ be two tangent vectors. Compute the lengths of $\vec{X}$, $\vec{Y}$, and the angle between them. Compute the second fundamental form $II(\vec{X}, \vec{Y})$.

5.6. (6pts) Define the Weingarten map L as a linear map from the tangent space to itself. Write down $\mathsf{L}(\mathbf{x}_1)$ and $\mathsf{L}(\mathbf{x}_2)$ in terms of $\mathbf{x}_1$ and $\mathbf{x}_2$.

5.7. (6pts) Compute the eight Christoffel symbols Γ_{ij}^k. You may use the extrinsic formula $\sum\langle \mathbf{x}_{ij}, \mathbf{x}_l\rangle g^{lk}$ or the intrinsic formula $\frac{1}{2}\sum\big(\frac{\partial g_{il}}{\partial u^j} + \frac{\partial g_{jl}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^l}\big)g^{lk}$.

5.8. (6pts) Consider the curves $u^1 = C$ and the curves $u^2 = D$, where C and D are constants, each parametrized by arc length. Compute their normal and geodesic curvatures (extrinsically or intrinsically, as you prefer). Which of these curves are geodesics?

5.9. (6pts) Compute the eigenvalues and eigenvectors of L. What are the principal curvatures and what are the lines of curvature?

5.10. (6pts) Compute the mean curvature and the Gaussian curvature. What are the integrals of the mean curvature and the Gaussian curvature over the whole surface T?