(Q-1) A real valued continuous function satisfies for all real x and y the functional equation

$$f(\sqrt{x^2+y^2}) = f(x)f(y).$$

Prove that $f(x) = [f(1)]^{x^2}$. [Hint: First prove the theorem for all numbers of the form $2^{n/2}$ where n is an integer. Then prove the theorem for all numbers of the form $\sqrt{m/2^n}$, m an integer, n a non-negative integer.]

(Q-2) Let f be defined in the interval $[0,1]$ by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational,} \\ 1/q & \text{if } x = p/q \text{ (in lowest terms).} \end{cases}$$

(a) Prove that f is discontinuous on each rational number in $[0,1]$.
(b) Prove that f is continuous on each irrational number in $[0,1]$.

(Q-3) Suppose $f\colon [0,1] \to [0,1]$ is continuous. Prove that there exists a number c in $[0,1]$ such that $f(c) = c$.

(Q-4) A rock climber starts to climb a mountain at 7am on Saturday and gets to the top at 5pm. He camps on top and climbs back down on Sunday, starting at 7am and getting back to his original starting point at 5pm. Show that at some time of day on Sunday he was at the same elevation as he was at that time on Saturday.

(Q-5) Define f by

$$f(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Let $g(x) = x + 2f(x)$. Show that $g'(0) > 0$ but that g is not monotonic in any open interval about 0.

(Q-6) (a) Show that $5x^4 - 4x + 1$ has a root between 0 and 1.
(b) If $a_0, a_1, \ldots, a_n$ are real numbers satisfying

$$\frac{a_0}{1} + \frac{a_1}{2} + \cdots + \frac{a_n}{n+1} = 0,$$

show that the equation $a_0 + a_1x + \cdots + a_nx^n = 0$ has at least one real root.

(Q-7) Evaluate $\lim_{n\to\infty} 4^n(1 - \cos(\theta/2^n))$.

(Q-8) Calculate

$$\lim_{x\to\infty} x \int_0^x e^{t^2-x^2}\, dt.$$

(Q-9) Evaluate each of the following:

(a) $\lim\limits_{n\to\infty} \left(\frac{n}{1^2+n^2} + \frac{n}{2^2+n^2} + \cdots + \frac{n}{n^2+n^2}\right)$.

(b) $\lim\limits_{n\to\infty} \left(\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\cdots\left(1+\frac{n}{n}\right)\right)^{1/n}$.

(Q-10) Let $f\colon [0,1] \to (0,1)$ be continuous. Show that the equation

$$2x - \int_0^x f(t)\, dt = 1$$

has one and only one solution in the interval $[0,1]$.